Automata, Computability, & Complexity

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Theory of Computation Overview

Main questions in theory of computation:

- What is computation? ⇒ *Automata* & *Turing Machines* as the mathematical model
- What *can* and *cannot* be computed? ⇒ *Computability*
- What can be *efficiently* computed? ⇒ *Complexity*

> Mathematical proof should provide 3 levels:
>
> 1. Short phrase giving "hints"
> 2. One paragraph description of main ideas
> 3. Full proof (and nothing else)

Automata

See my "Computer Languages & Compilers" course note. The following are something more formal.

Formal Definition of Language

An **Alphabet** Σ is a finite set of characters:

- A **String** s over Σ is a finite sequence of characters $\in \Sigma$
  - $|x| =$ length of string x
  - The unique string of length 0 (*empty* string) denoted by ε
- Σ^* is the set of all strings over Σ

A **Language** over Σ is a set of strings over Σ, i.e., a subset of Σ^*:

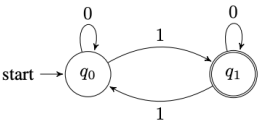
- Sometimes, we think of a language L as a *function* $f : \Sigma^* \mapsto \{0, 1\}$ (accepting a string or not?)
- Sometimes, we use $L(A)$ to denote set of all strings that an automaton A accepts

Operations on languages:

Notation	Meaning	Notes
$\emptyset$	Empty language	$\neq \{\varepsilon\}$
$L_1 \cup L_2$	Union	$\{w \mid w \in L_1 \vee w \in L_2\}$
$L_1 \cap L_2$	Intersection	$\{w \mid w \in L_1 \wedge w \in L_2\}$
$\neg L$ or $\bar{L}$	Complement	$\{w \in \Sigma^* \mid w \notin L\}$
L^R	Reverse	$\{w_1 \dots w_k \mid w_k \dots w_1 \in L, w_i \in \Sigma\}$
$L_1 L_2$	Concatenation	$\{w_1 w_2 \mid w_1 \in L_1 \wedge w_2 \in L_2\}$
L^*	0 or more self-concatenation (*Kleen star/closure*)	$\{w_1 \dots w_k \mid \text{each } w_i \in \Sigma \wedge k \geq 0\}$
L^+	1 or more self-concatenation	$\{w_1 \dots w_k \mid \text{each } w_i \in \Sigma \wedge k > 0\}$

Deterministic Finite Automata (DFA) & Regularity

A **DFA** is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, e.g.,



- Composed of:
  - Q: the finite set of *states*
  - Σ: the finite alphabet
  - δ: $Q \times \Sigma \mapsto Q$ is the *transition function*
  - $q_0 \in Q$ is the *start* state
  - $F \subseteq Q$ is a finite set of *accept/final* states
- M *accept*s a string $w = w_1 \dots w_n$ if there is a sequence $r_0, r_1, \dots, r_n \in Q$, s.t.: (o.w. called *reject*)
  - $r_0 = q_0$, and
  - $\delta(r_{i-1}, w_i) = r_i$ for all $i = 1, \dots, n$, and
  - $r_n \in F$

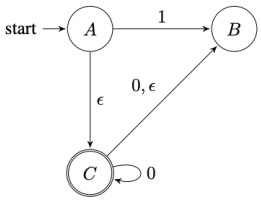
A language L' is **regular** iff L' is recognized by a DFA, that is, there is a DFA M where $L' = L(M)$.

Regular Languages (RLs) are *closed* under the following cases:

1. *Union Theorem*: Union of two RLs $L_1 \cup L_2$ is still regular; rule to construct DFA:
   - $Q = \{(q_1, q_2) \mid q_1 \in Q_1 \wedge q_2 \in Q_2\} = Q_1 \times Q_2$, i.e., composite pairs of states
   - $q_0 = (q_0^{(1)}, q_0^{(2)})$
   - $F = \{(q_1, q_2) \mid q_1 \in F_1 \vee q_2 \in F_2\}$
   - $\delta((q_1, q_2), c) = (\delta_1(q_1, c), \delta_2(q_2, c))$
2. *Intersection Theorem*: Intersection of two RLs $L_1 \cap L_2$ is still regular; rules:
   - same as unioned DFA, except that
   - $F = \{(q_1, q_2) \mid q_1 \in F_1 \wedge q_2 \in F_2\}$
3. *Complement Theorem*: Complement of an RL $\neg L$ is still regular; rules:
   - everything the same as L, except that
   - $F = \{q \in Q \mid q \notin F\}$, i.e., flip final & non-final states
4. *Reverse Theorem*: Reverse of an RL L^R is still regular; rules are not intuitive, we should first introduce NFAs
5. Similar for *Concatenation*, *Prefix*, & *Wrap*.

Non-deterministic Finite Automata (NFA)

An **NFA** is a 5-tuple $N = (Q, \Sigma, \delta, Q_0, F)$, e.g.,



- Composed of:

    - Q: the finite set of *states*
    - Σ: the finite alphabet
    - δ: $Q \times \{\Sigma \cup \varepsilon\} \mapsto 2^Q$ is the *transition function*, where 2^Q means the set of all possible subsets of Q
    - $Q_0 \subseteq Q$ is a set of *start* states
    - $F \subseteq Q$ is a finite set of *accept/final* states
- We allow multiple start states here, but normally the definition is only one; convertion is easy: add an extra state which stretchs ε-transitions to all the start states
- M *accept*s a string $w = w_1 \dots w_n$ if there is a sequence $r_0, r_1, \dots, r_n \in Q$, s.t.: (o.w. called *reject*)
    - $r_0 \in Q_0$, and
    - $r_i \in \delta(r_{i-1}, w_i)$ for all $i = 1, \dots, n$, and
    - $r_n \in F$

For every NFA N, there is a DFA M s.t. $L(M) = L(N)$:

- Idea: *Subset Construction*, setting $Q' = 2^Q$; **see my PL note - "Convert NFA → DFA"**
- This means a language L' is regular iff L' is recognized by an NFA!
- This means using NFAs in place of DFAs can make proofs about RLs much easier!

Constructing NFA over language operations: *Thompson's construction*; **see my PL note - "From RE → NFA"**.

Regular Expressions (REs) & Generalized NFA (GNFA)

First, **check my PL note - "Regular Expressions"**.

Expressiveness: DFA ≡ NFA ≡ RE. They all express regular languages. Proof is two fold, where RE ⇒ regular is easy. Regular ⇒ RE introduces the concept of GNFA.

A Generalized NFA (**GNFA**) can read in entire substrings on one edge (labeled by an RE), instead of only a single character, e.g.,

- Every NFA is also a GNFA, and
- Every RE represents a GNFA (with only two states)

Converting from an NFA N into a GNFA G with only two states, then we can prove that $L(N)$ (which is regular) is equivalent to the language represented by the RE on that only edge of G. Conversion goes as the following:

```
// Converting NFA to a two-states GNFA which is essentially an RE.
G convert(N) {
    if (#states == 2) return N;
    else {
        pick state q_rip which is not q_start or q_acc;
        for (every pair of states (q_i, q_j) that originally has path through q_rip)  // q_i can == q_j.
            R(q_i, q_j) += R(q_i, q_rip) R(q_rip, q_rip)^* R(q_rip, q_j);
        results in a 1-state fewer GNFA G';
        return convert(G');
    }
}
```

Non-Regular Languages

So far, our topics cover:

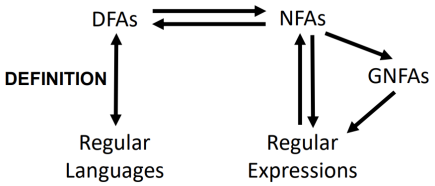


Figure from Ryan's slides.

There are many languages that are **non-regular** (Cannot find a DFA for it). A typical example is $L = \{0^n 1^n \mid n \geq 0\}$. The big idea here is that: A DFA cannot remember how many zeros it has read in, if there are more zeros than its # states. Its # states is finite!

More examples: $\{0^n 1^n \mid n \geq 0\}, \{w \mid w \text{ has equal number of 0s and 1s}\}, \{w \mid w = w^R\}$ (*palindromes*) ,

Proof method - The *confusion* technique: Assume there is a DFA M, and # states = Q. Carefully choose string x, where $|x| > Q$. Use *Pigeonhole Principle* to show that some state q must be visited at least twice. Then, M is in state q after reading x' or x'', where x' and x'' are two different prefixes of x. Then pick a string y s.t. exactly one $x'y$ or $x''y$ is in $L \Rightarrow$ *Contradiction*!

DFA Minimization

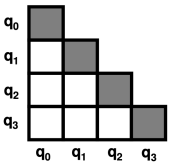
For every regular language L, there is a *unique* & *minimal* DFA M (minimal here means fewest number of states; unique here means up to re-labeling states) s.t. $L(M) = L$.

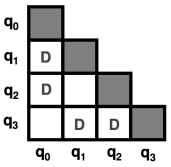
We introduce the following terms:

- For DFA $M = (Q, \Sigma, \delta, q_0, F)$, denote $M_q = (Q, \Sigma, \delta, q, F)$, $q \in Q$ which is the sub-DFA starting from q
- String $w \in \Sigma^*$ **distinguishes** states p and q iff: M_p accepts $w \Leftrightarrow M_q$ rejects w (i.e., leading to one accepting $\wedge$ the other rejecting)
  - If $\exists$ such w (including ε!), then $p \nsim q$ (*distinguishable* from)
  - If $\forall w \in \Sigma^*$, w cannot distinguish p from q, then $p \sim q$ (*indistinguishable* from)

Easy to prove that *indistinguishability* relation is an *equivalence relation*. This means we can partition Q into several *equivalence classes*, where each of them $[q] = \{p \mid p \sim q\}$. Big idea of minimizing DFA is to find such partition into equivalence classes, and shrinking each of them down to one state. Output M_{min} has the following three properties:

1. $L(M) = L(M_{min})$
2. M_{min} has no *inaccessible* states (i.e., unreachable from initial state)
3. M_{min} is *irreducible* (i.e., every pair of states is distinguishable)

Minimization algorithm: first prepare a table 

; (after the algorithm it looks like 

)

```
// Minimizing a DFA M.
M_min minimize(M) {
    Run a graph search from initial state q_0 to remove all inaccessible states from M;
    // Running the Table Filling algorithm...
    for (all (p, q) s.t. exactly one of them is an accepting state)
        Mark 'D' at entry (p, q);   // Base case: distinguishable by eps.
    while (there is update in this iteration) {     // Iter: infer pairs distinguishable with longer
strings.
        if (exists (p, q) and any symbol a in alphabet s.t. delta(p, a) = p2, delta(q, a) = q2, and
            we already marked 'D' at entry (p2, q2))
            Mark 'D' at entry (p, q);
    }
    Get EQUIV set = {[q] | q in Q} by those unmarked pairs;
    return M_min = (EQUIV, Sigma, delta_min([q], a) -> [delta(q, a)], [q_0], {[q] | q in F});
}
```

Detailed proof about correctness & uniqueness can be found in Lec5's slides.

The Myhill-Nerode Theorem

Let us define an equivalence relation which is purely on the language itself and not assuming any DFAs: Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$, $x \equiv_L y$ means that $\forall z \in \Sigma^*, xz \in L \Leftrightarrow yz \in L$. We call x and y are *indistinguishable to* L (or L-*equivalent*).

The **Myhill-Nerode Theorem** states that, a language L is regular $\Leftrightarrow$ the number of equivalence classes of $\equiv_L$ is finite.

- Proof is two-fold, by proving both directions; See Lec6's slides.
- Application: a new way of proving a given language is not regular - give an infinite set of strings S in L (called a *distinguishing set*) s.t. $\forall w_i \neq w_j \in S, w_i$ and w_j are *distinguishable* to L.

Streaming Algorithms

Streaming algorithms are an extension to finite automata. They have 3 components:

1. *Initialize*: a bunch of *variables* and their initial assignments
2. *Next symbol action*: pseudocode to operate on variables when seeing the next incoming symbol σ

3. *Stop*: accept/reject condition when the stream stops

They are very similar to DFAs except that their *memory usage* can increase with the string length. This gives them the ability to recognize non-regular languages!

Example - streaming algorithm that recognizes $L = \{x \mid x \text{ has more 1s than 0s}\}$ over alphabet $\{0, 1\}$:

```
// Initialize:
C = 0, B = 0;

// When next symbol is 'a':
if (C == 0) {
    B = a;
    C = 1;
} else if (C != 0 && B == a)
    C++;
else if (C != 0 && B != a)
    C--;

// When stream stops:
if (B == 1 && C > 0)
    ACCEPT;
else
    REJECT;
```

When using binary representation to store variables B and C, space usage of this algorithm is $\log_2 n + O(1)$.

Proving lower bounds of memory usage of a streaming algorithm:

- If language L is computable by a streaming algorithm with space usage $\leq S(n)$, then $\forall n \in \mathbb{N}$, there is a DFA M with $< 2^{S(n)+1}$ states s.t. $L_n = L(M)_n$, i.e., $\{w \in L \mid |w| \leq n\} = \{w \in L(M) \mid |w| \leq n\}$ (proof - see Lec7's slide);
- A *streaming distinguisher* D_n for L_n is a subset of Σ^* s.t. $\forall x, y \in D_n$, there is a string $z \in \Sigma^*$ s.t. $|xz|, |yz| \leq n$ and z distinguishes x, y;
- *Streaming Theorem*: Suppose $\forall n \in \mathbb{N}$, there is a streaming distinguisher D_n with $|D_n| \geq 2^{S(n)}$, then all streaming algorithms that recognize L must use at least $S(n)$ space.

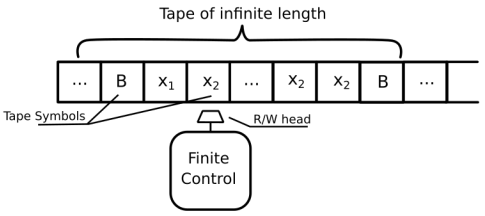
> For 2-pass streaming algorithm solution to FREQUENT ITEMS problem, see Lec7 slides.

Computability

Turing Machines (TM), Recognizability, & Decidability

Introduced by Alan Turing in 1936.

A **(Deterministic) Turing Machine** (TM) is defined as:



, in each step:

1. Reads a symbol
2. Writes a symbol
3. Finite control changes state
4. Moves left / right a block

A formal definition would be a 7-tuple $T = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$,

- Composed of:
  - Q: the finite set of states
  - Σ: the finite input alphabet
  - Γ: the finite tape alphabet, where $\square$ ("blank" symbol) $\in \Gamma$ and $\Sigma \subset \Gamma$
  - δ: $Q \times \Gamma \mapsto Q \times \Gamma \times \{L, R\}$, where L, R means moving left / right
  - $q_0 \in Q$ is the start state
  - $q_{acc} \in Q$ is the accepting state

    - $q_{rej} \in Q$ is the rejecting state, and $q_{acc} \neq q_{rej}$
- Initially, input string is written on tape and the control *head* points to the first symbol
- A TM *configuration* $\in (Q \cup \Gamma)^*$ is the stuff on tape with current state inserted right before control head
    - Configuration C_1 *yields* C_2 if T goes to C_2 after running for one step in C_1
    - T *accept*s a string w if there is a sequence of configurations s.t.: (reject is similar)
        - $C_0 = q_0 w$, and
        - C_{i-1} yields C_i for all $i = 1, \ldots, n$, and
        - C_n contains the accept state q_{acc}
- A TM can have 3 kinds of behaviors on an input string:
    1. *Accept*
    2. *Reject*
    3. Running forever w/o *halting*

Recognizability is defined as the following:

- A TM T *recognizes* L iff T accepts exactly the strings in L (but for other strings it might run forever)
- A language L is *recognizable* (i.e., *recursively enumerable*) if some T recognizes L

Decidability is defined as the following: (this is stronger than recognizability)

- A TM T *decides* L iff T accepts all strings in L, and rejects all strings not in L
- A language L is *decidable* (i.e., *recursive*) if some T decides L

Universal TM & The Church-Turing Thesis

We can encode a TM into a *bit string* $b = w_T w_{\delta_0} w_{\delta_1} \ldots$, where (DFA / NFA are similar)

- $w_T = 0^n 1 0^m 1 0^k 1 0^s 1 0^t 1 0^r 1 0^u 1$ encodes the machine metadata
    - $n =$ # states
    - $m =$ # tape symbols (we assume first k are input symbols)
    - $k =$ # input symbols
    - $s =$ index of the start state
    - $t =$ index of the accepting state
    - $r =$ index of the rejecting state
    - $u =$ index of the blank symbol
- Each $w_\delta = 0^p 1 0^i 1 0^q 1 0^j 1 (0+00) 1$ encodes a function map in $\delta : p, i \mapsto q, j, L/R$

We then encode a pair of bit strings $\langle b_1, b_2 \rangle$ as $0^{|b_1|} 1 b_1 b_2$. Then, any DFA / NFA / TM is just a language over $\Sigma = \{0, 1\}$!

- $A_{\text{DFA}} = \{\langle M, w \rangle \mid M \text{ encodes a DFA and } M \text{ accepts } w \in \Sigma^*\}$
- $A_{\text{NFA}} = \{\langle N, w \rangle \mid N \text{ encodes a NFA and } N \text{ accepts } w \in \Sigma^*\}$
- $A_{\text{TM}} = \{\langle T, w \rangle \mid T \text{ encodes a TM and } T \text{ accepts } w \in \Sigma^*\}$

A **Universal TM** U can simulate any other TMs, i.e., U accepts $\langle T, w \rangle \Leftrightarrow T$ accepts w. In other words, there is a TM that can run any other arbitrary TM code! Proof is in Lec9 slides.

- Since A_{DFA} is not regular, there isn't a DFA which can simulate any other DFA
- A_{DFA} and A_{NFA} are both decidable; A_{TM} is recognizable but undecidable

The **Church-Turing Thesis** states that "Any function on natural numbers can be calculated by an *effective method* (i.e., has an *algorithm*) $\Leftrightarrow$ it is computable by a TM". The Church-Turing Thesis implies that there are *unrecognizable* languages! We can prove by showing that there is no *onto* function from the set of all TMs ($\subseteq \{0,1\}^*$) to the set of all languages over $\{0,1\}$ (powerset of $\{0,1\}^*$, i.e., $2^{\{0,1\}^*}$). In other words, there are more problems to solve than there are programs to solve them.

The Halting Problem & Mapping Reduction

Example of a *recognizable* but *undecidable* problem:

- Consider the language $A_{\text{TM}} = \{\langle T, w \rangle \mid T \text{ encodes a TM and } T \text{ accepts } w \in \Sigma^*\}$. This is equivalent to the problem: given code of a TM T and an input w, does T accept w? This problem is recognizable but undecidable.
- Proof idea: *diagnolization* - suppose there is a machine U that decides A_{TM}, define a TM $D_U(T)$ that runs U on $\langle T, T \rangle$ and outputs the opposite answer. Now consider the input $\langle D_U \rangle$ for D_U: $D_U(\langle D_U \rangle)$ accepts $\Leftrightarrow U(\langle D_U, D_U \rangle)$ rejects $\Leftrightarrow D_U(\langle D_U \rangle)$ rejects. Contradiction!

Example of an *unrecognizable* problem:

- Consider the language $\neg A_{\text{TM}} = \{\langle T, w\rangle \mid T \text{ encodes a TM and } T \text{ does not accept } w \in \Sigma^*\}$ (w can be empty). This is equivalent to the "non-acceptance" problem of TMs.
- Lemma: L is decidable $\Leftrightarrow$ L and $\neg L$ are both recognizable.
- Then, $\neg A_{TM}$ is not recognizable, otherwise both A_{TM} and $\neg A_{TM}$ would be decidable.

The **Halting Problem** states that:

- Consider the language $\text{HALT}_{\text{TM}} = \{\langle T, w\rangle \mid T \text{ encodes a TM and } T \text{ halts on } w \in \Sigma^*\}$. This language is undecidable. In other words, there is no TM that can decide whether a TM halts on an input.
- Proof idea: suppose there is a TM U that decides HALT_{TM}, then we can use U to construct a TM T' that decides A_{TM}:
  - $T'(\langle M, w\rangle)$ runs $U(\langle M, w\rangle)$
  - If U rejects, then reject
  - If U accepts, run $M(w)$ and produce the output
- In general, this technique is: to prove language L is undecidable, we prove that "If L is decidable, then so is A_{TM}".

A function $f : \Sigma^* \mapsto \Sigma^*$ is a *computable* function if there is a TM T that halts with exactly $f(w)$ written on tape, for every input w. A language L_1 is **mapping reducible** to language L_2, formally $L_1 \leq_m L_2$, if there is a computable function f s.t. $\forall w \in \Sigma^*, w \in L_1 \Leftrightarrow f(w) \in L_2$. Here, f is called a *mapping reduction* (or *many-one reduction*) from L_1 to L_2.

- Mapping reducible relation $\leq_m$ is *transitive*
- If $L_1 \leq_m L_2$, then $\neg L_1 \leq_m \neg L_2$
- If $L_1 \leq_m L_2$ and L_2 is recognizable, then L_1 is recognizable; If $L_1 \leq_m L_2$ and L_1 is unrecognizable, then L_2 is unrecognizable
- If $L_1 \leq_m L_2$ and L_2 is decidable, then L_1 is decidable; If $L_1 \leq_m L_2$ and L_1 is undecidable, then L_2 is undecidable

> The language A_{TM} is "*complete*" in recognizability!

Interesting conclusions & corollaries, see Lec11 slides for details:

- $A_{TM} \leq_m \text{HALT}_{TM}$, $\neg A_{TM} \leq_m \neg\text{HALT}_{TM}$, $\text{HALT}_{TM} \leq_m A_{TM}$, $\text{HALT}_{TM} \equiv_m A_{TM}$
- $\text{EMPTY}_{TM} = \{\langle M\rangle \mid M \text{ is a TM s.t. } L(M) = \emptyset\}$, $\neg A_{TM} \leq_m \text{EMPTY}_{TM}$, thus unrecognizable
- $\text{EQ}_{TM} = \{\langle M, N\rangle \mid M, N \text{ are TMs and } L(M) = L(N)\}$, $\text{EMPTY}_{TM} \leq_m \text{EQ}$, thus unrecognizable

Oracle TMs & Turing Reduction

An **Oracle TM** T is one equipped with a set $B \subseteq \Gamma^*$ and an oracle tape, and T may enter a special state $q_?$ to ask queries about whether string on oracle tape $\in B$ or not (goes to q_{yes} if yes, o.w., q_{no}). Assume this "oracle" check finishes in one step.

A language L_1 is *decidable with* L_2 if there is an oracle TM with oracle L_2 that accepts strings in L_1 and rejects others. We say L_1 **Turing reduces** to L_2, formally $L_1 \leq_T L_2$.

- Turing reducible relation $\leq_T$ is *transitive*
- If $L_1 \leq_T L_2$ and L_2 is decidable, then L_1 is decidable; If $L_1 \leq_T L_2$ and L_1 is undecidable, then L_2 is undecidable

We claim that $A_{TM} \leq_T \text{HALT}_{TM}$:

- In other words, we can decide the acceptance problem, given an oracle for the halting problem
- Proof idea: write pseudocode for an oracle TM for A_{TM} on input $\langle M, w\rangle$:
  - If $\langle M, w\rangle \in \text{HALT}_{TM}$, then run $M(w)$ and outputs its answer
  - Else, reject

Interesting conclusions & corollaries, see Lec11 slides for details:

- $A_{TM} \leq_T \text{HALT}_{TM}$, $\text{HALT}_{TM} \leq_T A_{TM}$, $\text{HALT}_{TM} \equiv_T A_{TM}$
- If $L_1 \leq_m L_2$, then $L_1 \leq_T L_2$
- $\neg A_{TM} \leq_T A_{TM}$, $\neg A_{TM} \not\leq_m A_{TM}$

Oracle TMs cannot solve all problems. In fact, there is an infinite hierarchy of unsolvable problems:

1. $\text{SUPERHALT}^0 = \text{HALT} = \{\langle M, x\rangle \mid M \text{ halts on } x\}$
2. $\text{SUPERHALT}^1 = \{\langle M, x\rangle \mid M, \text{with oracle for HALT, halts on } x\}$
3. $\text{SUPERHALT}^2 = \{\langle M, x\rangle \mid M, \text{with oracle for } \text{SUPERHALT}^1, \text{halts on } x\}$
4. ...

Self-Reference & Recursion

There is a computable function $q : \Sigma^* \mapsto \Sigma^*$ s.t., for every string w, $q(w) = \langle P_w\rangle$ where P_w always prints out w and then accepts. Based on this, we can define a self-printing TM by:

- $B(\langle M\rangle) = \langle$ "takes input w, outputs $M(\langle M\rangle)$" $\rangle$
- Now consider the TM which takes input w and runs $B(\langle B\rangle)$: it just prints out a machine description $\langle$ "takes input w, outputs $B(\langle B\rangle)$" $\rangle$

> This is the essential idea behind *quine programs*.

More formally, the *Recursion Theorem* states that for every computable function $T : \Sigma^* \times \Sigma^* \mapsto \Sigma^*$, there $\exists$ TM $R : \Sigma^* \mapsto \Sigma^*$, such that for every string w, $R(w) = T(\langle R\rangle, w)$. Construction of such TM:

- $B(\langle M\rangle) = \langle$ "takes input w, outputs $M(\langle M\rangle, w)$" $\rangle$
- $Q(\langle M\rangle, w) = T(B(\langle M\rangle), w)$
- $R(w) = T(B(\langle Q\rangle), w)$, we can show that $B(\langle Q\rangle) \equiv \langle R\rangle$

This theorem implies that we can use the operation "obtain your own description" in TM pseudocode!

A novel approach to prove that A_{TM} is undecidable: Assume $H(\langle M\rangle, w)$ decides A_{TM}. Define $T(\langle M\rangle, w)$ to run H and outputs the opposite answer. According to the Recursion Theorem, there is a TM R s.t. $R(w) = T(\langle R\rangle, w)$ and it would say R rejects w when R accepts w. Contradiction.

Also see proof of $\text{MIN} = \{\langle M\rangle \mid M \text{ is minimal-state}\}$ is undecidable in Lec12 slides.

Formal Systems

See Lec12 slides page 17~25. I don't really catch them.

Complexity

Communication Complexity

Consider a theoretical model of *communication*: we have a function $f(x, y) : \{0,1\}^* \times \{0,1\}^* \mapsto \{0,1\}$, where

- Alice only knows x, a binary string input; Bob only knows y, another binary string input
- Goal: get result of $f(x, y) = 0/1$? by communicating as few bits as possible between Alice & Bob
- Assume they alternate in communicating, and in every step send 1 bit / a STOP signal, and the last bit sent is the result

Define a **protocol** computing f to be a pair of functions $A, B : \{0,1\}^* \times \{0,1\}^* \mapsto \{0, 1, \text{STOP}\}$, with the semantics:

```
// Communication protocol semantics
r = 0, b_r = eps;
while (b_r != STOP) {
    r++;
    if (r is odd)
        Alice sends b_r = A(x, b_1...b_r-1);
    else
        Bob   sends b_r = B(y, b_1...b_r-1);
}
output f(x, y) = b_r-1;
```

The *cost* of a communication protocol (A, B) on n-bit strings is the *maximum* number of *rounds* (i.e. $r - 1$) taken by the algorithm over all possible input $x, y \in \{0,1\}^*$. The *communication complexity* of f on n-bit strings, $cc(f)$, is the *minimum* cost over all protocols computing f on n-bit strings.

- We are only interested in the communication cost, not including computation on either side
- There is always a trivial protocol for any f:
  - Alice sends the bits of x in odd rounds and Bob sends the bits of y in even rounds
  - $\Rightarrow$ For every function f, $cc(f) \leq 2n$
- Connection to streaming algorithms & DFAs: For x, y with $|x| = |y|$, define $f_L(x, y) = 1 \Leftrightarrow xy \in L$
  - If L has a streaming algorithm using $\leq S(n)$ space on string length $\leq 2n$, then $cc(f_L) \leq O(S(n))$. Idea: Alice runs streaming algorithm on x and reaches a memory configuration, she sends that configuration to Bob in $O(S(n))$ rounds, and Bob continues the algorithm on y from that configuration, then sends result bit to Alice
  - $\Rightarrow$ For every regular L, it has a DFA to compute, so $cc(f_L) = O(1)$

Proving lower bounds of communication complexity:

- Define *communication pattern* of a protocol on input x, y to be the sequence of bits Alice & Bob send, i.e. $b_1 b_2 \dots b_{r-1}$. If x, y and x', y' have the same pattern P, then x, y' and x', y also have pattern P
- Suppose $cc(f) \leq n - 1$, then there are $\leq 2^n - 1$ possible patterns. By the Pigeonhole Principle, ...

Computation Time Complexity

The very basic notations of *asymptotic complexity* (*Bachman-Landau notations*):

Notation	Definition	Abbr.	Meaning
$f(n) = o(g(n))$	$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$	$<$	f is dominated by g (strictly "smaller" than g)
$f(n) = O(g(n))$	$\lim_{n\to\infty} \lvert\frac{f(n)}{g(n)}\rvert < \infty$	$\leq$	f is bounded above by g
$f(n) = \Theta(g(n))$	$f(n) = O(g(n)) \wedge f(n) = \Omega(g(n))$	$=$	f is "equal" to g
$f(n) = \Omega(g(n))$	$\lim_{n\to\infty} \lvert\frac{f(n)}{g(n)}\rvert > 0$	$\geq$	f is bounded below by g
$f(n) = \omega(g(n))$	$\lim_{n\to\infty} \lvert\frac{f(n)}{g(n)}\rvert = \infty$	$>$	f dominates g (strictly "larger" than g)

Measuring worst-case **time complexity** of a TM can be done on counting the steps taken for a TM to halt on input of length n. Formally, $T : \mathbb{N} \mapsto \mathbb{N}$ where $T(n) =$ the maximum number of steps taken by M over all inputs of length n.

Example of a TM for deciding $L = \{0^k 1^k \mid k \geq 0\}$ using $O(n \log n)$ time (we say $L \in \mathrm{TIME}(n \log n)$):

1. If w is not of the form 0^*1^*, reject
2. Repeat untill all bits crossed out:
    1. If parity of 0s $\neq$ parity of 1s, reject
    2. Cross out every other 0; Cross out every other 1
    3. If all bits are crossed out, accept

To prove that there is no algorithm using less time, we can prove that any $\mathrm{TIME}(\frac{n \log n}{\alpha(n)})$, $\alpha(n)$ is an unbounded function, it contains only regular languages, thus not including L.

Different computation models can yield different time complexity! For example, on a two-tapes TM, there is an $O(n)$ algorithm for L:

1. Sweep over all 0s, copy them onto the second tape
2. Sweep over all 1s, each time crossing out a 0 from the second tape

Every multi-tape TM using $t(n)$ time on a language L has an equivalent $O(t^2(n))$ time one-tape TM $\Rightarrow$ Language decidable in polynomial time on any multi-tape TM can be decided in polynomial time on a traditional one-tape TM. See an intuitive simulation on Lec13 slides, page 17-20.

An *efficient* universal TM U is one that takes in an extra input 1^t and simply rejects when the simulated TM exceeds t steps. Obviously, U accepts $\langle M, w, 1^t \rangle \Leftrightarrow M$ accepts w in t steps. We can also guarantee that such U runs in $O(|M|^2 t^2)$ time.

The **Time Hierarchy Theorem** states that for all reasonable $f, g : \mathbb{N} \mapsto \mathbb{N}$ where $g(n) > n^2 f^2(n)$ for all n, $\mathrm{TIME}(f(n)) \subsetneq \mathrm{TIME}(g(n))$. "We can solve *strictly more* problems if given quadratically more time to compute."

- Proof idea is to use *diagonalization* with a clock, and the contradiction implies that the input cannot be decided in $f(|M|)$ time. But, by picking $g(|M|) > |M|^2 t^2 = |M|^2 f^2(|M|)$, we can construct a universal TM running in $O(g(|M|))$ time to simulate a TM over this problem that runs in $f(|M|)$ time
- Claim actually still holds for $g(n) > f(n) \log^2 f(n) \Rightarrow \mathrm{TIME}(n) \subsetneq \mathrm{TIME}(n^2) \subsetneq \mathrm{TIME}(n^3) \subsetneq \ldots$

Nondeterminism & P vs. NP

Define:

$$\mathrm{P} = \bigcup_{k \in \mathbb{N}} \mathrm{TIME}(n^k).$$

- $\mathrm{PRIMES} = \{n \mid n \text{ is a prime number}\} \in \mathrm{P}$

These are the *effectively decidable* problems in the world of complexity theory (can be efficiently solved on real-world machines). The **Extended Church-Turing Thesis** states that our notion of *efficient* algorithms $\Leftrightarrow$ polynomial time TMs. This thesis is under doubt nowadays because of the existense of quantum algorithms.

Define a *decidable predicate* $R(x, y)$ as a proposition about inputs x, y s.t. some TM M *implements* R (i.e., $R(x, y) = \text{true} \Rightarrow M(x, y)$ accepts, $R(x, y) = \text{false} \Rightarrow M(x, y)$ rejects; i.e., R is a computable function: $\Sigma^* \times \Sigma^* \mapsto \{\text{true}, \text{false}\}$). Theorems states that a language A is recognizable $\Leftrightarrow$ there is a decidable predicate $R(x, y)$ s.t. $A = \{x \mid \exists y \in \Sigma^*, R(x, y) = \text{true}\}$.

- This bridges recognizability via decidability. The proof is trivial for $\Rightarrow$, and for $\Leftarrow$, let $R(x, y)$ be true iff M accepts x in $|y|$ steps.
- Examples of using this theorem to show some language is recognizable - $L = \{\langle M \rangle \mid M \text{ accepts at least one string}\}$: let $R(\langle M \rangle, \langle x, y \rangle)$ be true iff M accepts string x in $|y|$ steps. Keep guessing some x and *verify* it in finite $|y|$ time!
- This implies a very important corollary: "**Determinism** vs. **Nondeterminism**, are they equally powerful?"
  - Yes for finite automata
  - No for Turing machines
  - ??? for polynomial time

A **Nondeterministic Turing Machine** (NTM) is one whose state diagram is an NFA. A formal definition is a 7-tuple $N = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$,

- Composed of:
  - Normal components of a TM
  - $\delta : Q \times \Gamma \mapsto 2^{Q \times \Gamma \times \{L,R\}}$
- An *accepting computation history* for N on w is a sequence of configurations $C_0, C_1, \ldots, C_t$ following the definition of acceptance of a TM
  - $N(w)$ accepts in t time $\Leftrightarrow$ such a history exists
  - N has *time complexity* $T(n)$ if on all inputs w of length at most n, N halts in $T(n)$ time

Now we can define $\text{NTIME}(t(n)) = \{L \mid L \text{ is decided by an } O(t(n)) \text{ time NTM}\}$. Define:

$$\text{NP} = \bigcup_{k \in \mathbb{N}} \text{NTIME}(n^k).$$

Can think of it as "recognizability" in complexity field. It means that tracing down a specific computation history in the "nondeterministic parallel tree" takes polynomial time. Equivalently, it means there is an algorithm which can *verify* (i.e., *prove*) whether a given solution (*certificate*) is correct, in polynomial time. [Existential Analogy]

Formally, $L \in \text{NP} \Leftrightarrow$ there is a constant k and a polynomial-time NTM V (verifier) s.t.
$L = \{x \mid \exists y \in \Sigma^* \text{ where } |y| \leq k|x|^k, V(x, y) = \text{true}\}$.

Examples of NP problems:

- Boolean SAT problems: $\text{SAT} = \{\phi \mid \phi \text{ is a satisfiable boolean formula}\}$
  - Special case: *conjunctive negation formula* (CNF) - conjunction of clauses who are disjunctions of literals
  - 3-CNF takes the form: $(x_1 \vee \neg x_2 \vee x_3) \wedge (x_4 \vee x_2 \vee \neg x_5) \wedge \cdots$; each clause has 3 literals
  - $\text{3-SAT} = \{\phi \mid \phi \text{ is a satisfiable 3-CNF}\}$
- Hamiltonian path problem
- EQUIV / NEQUIV formula
- K-Clique problem; Independent set; Vertex cover
- Knapsack problems (0-1)
- ... (many more)

NP-Completeness & The Cook-Levin Theorem

A **polynomial time reduction** is a mapping reduction where the function f is computable in polynomial time. We denote that as $A \leq_P B$.

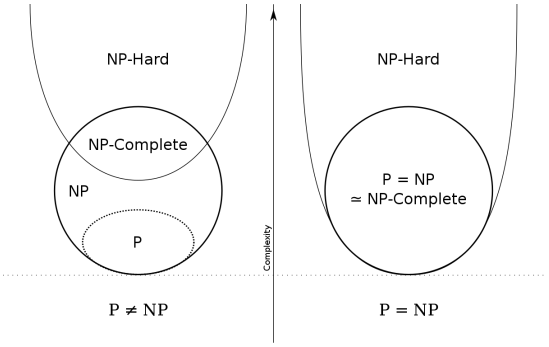
- A useful property of such f is that for any input w, $|f(s)| \leq k|w|^k$
- $\leq_P$ is also transitive
- If $L_1 \leq_P L_2$ and $L_2 \in \text{P}$, then $L_1 \in \text{P}$; If $L_1 \leq_P L_2$ and $L_1 \notin \text{P}$, then $L_2 \notin \text{P}$
- If $L_1 \leq_P L_2$ and $L_2 \in \text{NP}$, then $L_1 \in \text{NP}$; If $L_1 \leq_P L_2$ and $L_1 \notin \text{NP}$, then $L_2 \notin \text{NP}$

So we can ask: what are the "hardest" NP problems under such reduction? We define a language L to be **NP-Complete** (NPC) if $L \in \text{NP}$ and L is **NP-Hard**: for every $L' \in \text{NP}$, $L' \leq_P L$.

- We can easily show that $A_{NTM} = \{\langle N, w, 1^t \rangle \mid \text{ Nondeterministic TM } N \text{ accepts } w \text{ in } \leq t \text{ steps}\}$ is NPC
- The **Cook-Levin Theorem** states that the 3-SAT language is NPC! See Lec15 slides for a high-level proof.

> The entire P=NP? question can be answered if we can prove whether a logic problem $\text{3-SAT} \in$ or $\notin \text{P}$.

3-SAT can be polynomial-time reduced to problems including (see Lec16 slides): Clique, Vertex cover, Independent set, Subset sum, Knapsack, Fair partition, Bin packing, Hamiltonian path, Longest path.... So these are all NP-Complete, thus equivalent in some deep sense.



CoNP & Oracle Complexity

The class **coNP** is the set of languages whose complement is in NP: $\text{coNP} = \{L \mid \neg L \in \text{NP}\}$. This is called *conondeterministic computation*. For a coNP language L, there is a polynomial-time algorithm that can verify that a certificate is not in L (i.e., can efficiently verify a counter-example). In other words, a conondeterministic machine "tries all" polynomial-time paths and accepts only if all these paths lead to accept. [Universal Analogy]

- $\text{P} \subseteq \text{coNP}$; generally, deterministic complexity is closed under complement, so in fact $L \in \text{P} \Leftrightarrow \neg L \in \text{P}$
- Is $\text{NP} = \text{coNP}$? It is also an open question!

Examples of coNP problems:

- $\text{TAUTOLOGY} = \{\phi \mid \text{every variable assignment satisfies } \phi\} = \{\phi \mid \neg\phi \in \text{UNSAT}\}$
- $\text{UNSAT} = \{\phi \mid \text{no variable assignment can satisfy } \phi\} = \neg\text{SAT}$
- ...

A language is **coNP-Complete** if $L \in \text{coNP}$ and L is **coNP-Hard**: for every $L' \in \text{coNP}$, $L' \leq_P L$.

- Key trick: $A \leq_P B \Leftrightarrow \neg A \leq_P \neg B$, so we can easily prove coNP-completeness by using NP-completeness
- Easy to see that UNSAT is coNP-complete, and the same for others

A very important class of languages is the $\text{NP} \cap \text{coNP} = \{L \mid L \in \text{NP} \wedge \neg L \in \text{NP}\}$.

- $\text{FACTORING} = \{(n.\, k) \mid n > k > 1 \text{ and there is a prime factor } p \text{ of } n \text{ where } k \leq p < n\} \in \text{NP} \cap \text{coNP}$
- Is $\text{P} = \text{NP} \cap \text{coNP}$? is also an open question!

Next we move on to *complexity classes with oracles*. Denote P^B be the set of languages decidable in polynomial-time with an oracle for B. Similarly:

- P^P be the class of languages decidable in polynomial-time with an oracle for some language in P
  - $\text{P}^\text{P} = \text{P}$ because running polynomial steps of polynomials still gives a polynomial
- P^NP be the class of languages decidable in polynomial-time with an oracle for some language in NP
  - For any language B which is NP-complete, $\text{P}^B = \text{P}^\text{NP}$; For example, $\text{P}^\text{NP} = \text{P}^\text{SAT} = \cdots$
  - $\text{NP} \subseteq \text{P}^\text{NP}$, $\text{coNP} \subseteq \text{P}^\text{NP}$, $\text{P}^\text{NP} = \text{P}^\text{coNP}$
  - $\text{FIRST-SAT} = \{(\phi, i) \mid \phi \text{ is satisfiable and } i\text{-th variable of the lexicographically first satisfying assignment is true}\} \in \text{P}^\text{NP}$
- NP^NP be the class of languages decidable by a polynomial-time NTM with an oracle for some language in NP
  - Is $\text{NP} = \text{NP}^\text{NP}$? is an open question!
- coNP^NP be the class of languages decidable by a polynomial-time conondeterministic TM with an oracle for some language in NP
  - Is $\text{coNP}^\text{NP} = \text{NP}^\text{NP}$? is an open question!
  - $\text{MIN-FORMULA} = \{\phi \mid \phi \text{ is minimal among all its equivalences}\} \in \text{coNP}^\text{NP}$

Space Complexity

We measure **space complexity** by finding the largest tape index reached during computation of TM. The worst-case space complexity $S(n)$ is the largest tape index reached by TM on any input of length n.

Use $\text{SPACE}(s(n))$ as the set of languages decided by a TM with $O(s(n))$ space complexity. We have $\text{NTIME}(t(n)) \subseteq \text{SPACE}(t(n))$. Every multi-tape TM using $s(n)$ space on a language L has an equivalent $O(s(n))$ space one-tape TM.

Like the Time Hierarchy, we have the **Space Hierarchy Theorem** that for all reasonable $f, g : \mathbb{N} \mapsto \mathbb{N}$ where $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, $\text{SPACE}(f(n)) \subsetneq \text{SPACE}(g(n))$. "We can solve *strictly more* problems if given more space to compute."

Define:

$$\mathrm{PSPACE} = \bigcup_{k \in \mathbb{N}} \mathrm{SPACE}(n^k).$$

- $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE}$
- Intuition: You can always re-use space, but you cannot re-use time! So same order of space is intuitively stronger.

For every *halting* TM using $s(n)$ space, the upper bound of running time is the total number of possible configurations $2^{O(s(n))}$, otherwise the TM loops. This implies that there is a TM running in $2^{O(s(n))}$ time that decides the same language.

This means that if we define:

$$\mathrm{EXPTIME} = \bigcup_{k \in \mathbb{N}} \mathrm{TIME}(2^{n^k}).$$

- Then $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXPTIME}$
- And $\mathrm{NP}^{\mathrm{NP}}$ and $\mathrm{coNP}^{\mathrm{coNP}} \subseteq \mathrm{PSPACE}$, thus $\subseteq \mathrm{EXPTIME}$

We can also have nondeterminism here. Define $\mathrm{NSPACE}(s(n))$ as the set of languages decided by an NTM with $O(s(n))$ space complexity. Define:

$$\mathrm{NPSPACE} = \bigcup_{k \in \mathbb{N}} \mathrm{NSPACE}(n^k).$$

- $\mathrm{NPSPACE}$ is also $\subseteq \mathrm{EXPTIME}$
- The **Savitch's Theorem** states that for functions $s(n) \geq n$, $\mathrm{NSPACE}(s(n)) \subseteq \mathrm{SPACE}(s^2(n))$, this means $\mathrm{PSPACE} = \mathrm{NPSPACE}$!

One example of PSPACE-complete problems is: True-Fully-Quantified-Boolean-Formula (TQBF).

> P, NP, & PSPACE complexity can model the complexity of some games:
>
> - P captures short "zero-player" games, such as *Conway's Game of Life*
> - NP captures short "one-player" games with a goal, including many single-perspective video games
> - PSPACE captures short "two-player" games with a winning strategy

Randomized Complexity

A **probabilistic TM** M is an NTM where each nondeterministic step is a *coin flip*. Suppose each step has only two legal next moves. The probability that M runs on a path p is $Pr[p] = 2^{-k}$, where k is the number of coin flips that occur on p.

The probability that M accepts input w is $Pr[M \text{ accepts } w] = \sum_{p \text{ accepts on } w} Pr[p]$.

- Language L in NP means $w \in L \Leftrightarrow Pr[M \text{ accepts } w] > 0$
- Language L in coNP means $w \notin L \Leftrightarrow Pr[M \text{ accepts } w] > 0$

A probabilistic TM M decides a language L *with error* ε iff for all input w:

- $w \in L \Rightarrow Pr[M \text{ accepts } w] \geq 1 - \varepsilon$, and
- $w \notin L \Rightarrow Pr[M \text{ doesn't accept } w] \geq 1 - \varepsilon$.

We can define *Bounded Probabilistic* P (BPP) to be the set of languages decided by some probabilistic polynomial-time TM with error at most $\varepsilon \leq \frac{1}{2} - \frac{1}{n^k}$ for some k. Using the *Error Reduction Lemma* as stated in Lec21 slides, we can show that for any language L in BPP, there is an equivalent machine M' which decides L in the same time complexity with error $< \frac{1}{2^{n^{2k}}}$. A one-sided version is *Randomized* P (RP) where negative inputs are always not accepted but positive inputs have bounded error.

One example of BPP problems is: Zero-Identical-Polynomial (ZERO-POLY). It is not known how to solve ZERO-POLY efficiently without randomness.

> A widely-conjectured complexity space venn diagram looks like the following:

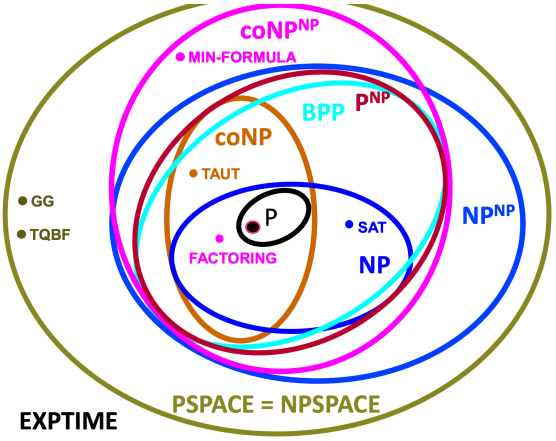


Figure taken from Ryan's slides.