Introduction to Algorithms

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Full-ver. Cheatsheet

See below (page 2-5).

Midterm Review Notes (TeX)

See below (page 6-).

THANK YOU, ANNIE, FOR THE WONDERFUL NOTES! ❤

$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

$\sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$

* Interval Scheduling: earliest finishing time.

* Interval Coloring: smallest available resource (d)

* Huffman: freq最小两个加起来 → new node.

Greedy

* Maximum Subarray: L+R+crossing (L+R)

* Closest point pair: L+R+crossing (<δ)
sort by y (crossing)
check 5 points below each point.

D&C

$$\log n! = n \log n - (\log e)n + O(\log n)$$

$$T(n) = aT(n/b) + f(n)$$

1. If $f(n) = O(n^{\log_b a - \epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$, then $T(n) = \Theta(f(n))$.

* insertion sort: $O(n^2)$ stable

* Quicksort: average O(nlogn) unstable

```
PARTITION(A, p, r)
1  x = A[r]
2  i = p − 1
3  for j = p to r − 1
4      if A[j] ≤ x
5          i = i + 1
6          exchange A[i] with A[j]
7  exchange A[i + 1] with A[r]
8  return i + 1
```
O(n)

```
COUNTING-SORT(A. B, k)
1  let C[0 . . k] be a new array
2  for i = 0 to k
3      C[i] = 0
4  for j = 1 to A.length
5      C[A[j]] = C[A[j]] + 1
6  for i = 1 to k
7      C[i] = C[i] + C[i − 1]
8  for j = A.length downto 1
9      B[C[A[j]]] = A[j]
10     C[A[j]] = C[A[j]] − 1
```
* stable Θ(n+k).
O(n) if k=O(n)
* radix sort: O(d(n+k)) stable

```
MAX-HEAPIFY(A, i)
1  l = 2i, r = 2i + 1
2  if l ≤ A.heap-size and A[l] > A[i]
3      largest = l
4  else
5      largest = i
6  if r ≤ A.heap-size and A[r] > A[largest]
7      largest = r
8  if largest ≠ i
9      exchange A[i] with A[largest]
10     MAX-HEAPIFY(A, largest)
```
O(logn)

```
BUILD-MAX-HEAP(A)
1  A.heap-size = A.length
2  for i = ⌊A.length/2⌋ downto 1
3      MAX-HEAPIFY(A, i)
```
Complexity: $O(n)$

```
HEAPSORT(A)
1  BUILD-MAX-HEAP(A)
2  for i = A.length downto 2
3      exchange A[1] with A[i]
4      A.heap-size = A.heap-size − 1
5      MAX-HEAPIFY(A, 1)
```
unstable. O(nlogn)

* binary search: O(logn)

```
TERNARY-SEARCH(f, l, r, ϵ)
1  if r − l < ϵ
2      return (r + l)/2
3  x = l + (r − l)/3, y = r − (r − l)/3
4  if f(x) < f(y)
5      return TERNARY-SEARCH(f, l, y, ϵ)
6  if f(x) > f(y)
7      return TERNARY-SEARCH(f, x, r, ϵ)
8  return TERNARY-SEARCH(f, x. y. ϵ)
```

```
MEDIAN-OF-MEDIANS(A, p, r)
1  l = ⌈(r − p + 1)/5⌉
2  let M[1 . . l] be a new array
3  for k = 1 to l − 1
4      M[k] = A[SELECT(A, 5k + p − 5,
           5k + p − 1, 3)]
5  M[l] = A[SELECT(A, 5l + p − 5, r,
           ⌈(r − 5l − p + 6)/2⌉)]
6  m = SELECT(M, 1, l, ⌊l/2⌋)
7  if m == l
8      return ⌊(5l + p − 5 + r)/2⌋
9  return 5m + p − 3
```

* randomized select: expected O(n)

```
SELECT(A, p, r, i)
1  if r − p + 1 ≤ 5
2      INSERTION-SORT(A[p . . r])
3      return p − 1 + i
4  exchange A[r] with
       A[MEDIAN-OF-MEDIANS(A, p, r)]
5  q = PARTITION(A, p. r)
6  k = q − p + 1
7  if i == k
8      return q
9  elseif i < k
10     return SELECT(A, p. q − 1, i)
11 else
12     return SELECT(A, q + 1, r, i − k)
```
S(n) = S(n/5) + S(n) + O(n)

Complexity: $O(n)$

To multiply binary numbers a and b. split their bits in half, $a = 2^{n/2} \cdot a_1 + a_0$, $b = 2^{n/2} \cdot b_1 + b_0$, then $ab = 2^n \cdot a_1 b_1 + 2^{n/2} \cdot (a_1 b_0 + a_0 b_1) + a_0 b_0 = 2^n \cdot a_1 b_1 + 2^{n/2} \cdot ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0) + a_0 b_0$.

* FFT: $\omega_n = e^{\frac{2\pi i}{n}}$ $A^{[0]} = a_0 + a_2 x + \cdots a_{n-2} x^{\frac{n}{2}-1}$ $A^{[1]} = a_1 + a_3 x + \cdots a_{n-1} x^{\frac{n}{2}-1}$

```
BST-FIND(x, k)
1  if x == NIL or k == x.key
2      return x
3  if k < k.key
4      return FIND(x.left. k)
5  return FIND(x.right, k)
```
$A(x) = A^{[0]}(x^2) + xA^{[1]}(x^2)$
$\{\omega_n^0, \cdots \omega_n^{n-1}\} \to \{\omega_{n/2}^0, \cdots \omega_{n/2}^{\frac{n}{2}-1}\}$

```
BST-SUCCESSOR(x)
1  if x.right ≠ NIL
2      return left most node in x.right
3  y = x.p
4  while y ≠ NIL and x == y.right
5      x = y, y = y.p
6  return y
```
O(nlogn)
$a_j = \frac{1}{n} \sum_{k=0}^{n-1} y_k \omega_n^{-kj}$

```
COUNT-CROSSING-INVERSIONS(A, p, q, r)
1  n1 = q − p + 1
2  n2 = r − q
3  let L[0 . . n1 − 1] and R[0 . . n2 − 1]
       be new arrays
4  num = 0
5  for i = 0 to n1 − 1
6      L[i] = A[p + i]
7  for j = 0 to n2 − 1
8      R[j] = A[q + 1 + j]
9  i = 0
10 j = 0
11 k = p
12 while i < n1 and j < n2
13     if L[i] > R[j]
14         num = num + 1
15         A[k] = R[j]
16         j = j + 1
17     else
18         num = num + j
19         A[k] = L[i]
20         i = i + 1
21     k = k + 1
22 if j == n2
23     num = num + (n1 − i − 1) ∗ n2
24     for r = 0 to n1 − i − 1
25         A[k + r] = L[i + r]
26 else
27     num = num − j
28     for r = 0 to n2 − j − 1
29         A[k + r] = R[j + r]
30 return num
```
* mergesort: stable.
O(nlogn). 2n memory
L[n1] = R[n2] = ∞
* merge 末端判断
L+R+crossing. O(nlogn)

```
BST-INSERT(T, k)
1  x = T.root
2  if FIND(x, k) == NIL
3      FIND(x, k) reaches node y
4      if y == NIL (empty tree)
5          insert in y
6      elseif z.key < y.key
7          insert in y.left
8      else
9          insert in y.right
```

```
BST-DELETE(T, z)
1  if z.left == NIL
2      TRANSPLANT(T, z, z.right)
3  elseif z.right == NIL
4      TRANSPLANT(T, z, z.left)
5  else
6      y = MINIMUM(z.right)
7      if y.p ≠ z
8          TRANSPLANT(T, y, y.right)
9          y.right = z.right
10         y.right.p = y
11     TRANSPLANT(T, z, y)
12     y.left = z.left
13     y.left.p = y
```
* transplant.
交换指针. O(1)

* insert. delete. extract-max.
O(logn) ⇒ priority queue.

* tips: log → D&C, 二分答案. a.b 排序. 区间b−a. x.÷有意义时尝试.
最→D&C / greedy greedy证明: ① 至少写同其它算法. ② 约束条件 (interval coloring)
图论复杂度低→DFS. timestamp. ③ 其它最优解⇒贪心解.

* Rotation: 交换指针. O(1)

* Heuristic hash functions:

Division method: $h(k)=k \bmod m$. m prime, not close to 2^t.

Multiplication method: $h(k)=\lfloor m(kA \bmod 1)\rfloor$, $A\in(0,1)$

AVL-Insert: BST-INSERT first, then find the **lowest unbalanced node** and use $O(1)$ (one or two) rotations to restore balance.
AVL-Delete: BST-DELETE first, then **bottom up** find unbalanced node and use $O(h)$ rotations to restore balance.
O(logn)

Hash table $h: U \to \{0,1,\dots,m-1\}$
Call $\alpha = n/m$ **load factor**. The average time for each operation is $O(\alpha)$.
Universal hash family H: $\forall$ keys $x \neq y$, $P_{h\in H}[h(x)=h(y)] \le 1/m$.
Constructing: Choose a prime number p such that $p > m$, and all keys $< p$. $h_{ab}(k) = ((ak+b) \bmod p) \bmod m$. $H_{pm} = \{h_{ab} \mid a \in \{1,2,\dots,p-1\}, b \in \{0,1,\dots,p-1\}\}$

Open addressing
Linear probing: $h(k,i) = (h'(k)+i) \bmod m$. → m probe sequences
Quadratic probing: $h(k,i) = (h'(k)+c_1 i + c_2 i^2) \bmod m$.
Double hashing: $h(k,i) = (h_1(k) + i h_2(k)) \bmod m$. make m prime. $m' = m-1$, and set $h_1(k) = k \bmod m$, $h_2(k) = 1 + (k \bmod m')$. / $m=2^t$, h_2 odd. $O(m^2)$ probe sequences
$h: U \times \{0,\dots,m-1\} \to \{0,\dots,m-1\}$

BFS(G,s)
```
1  for each vertex u ∈ G.V − {s}
2      u.color = WHILE
3      u.d = ∞
4      u.π = NIL
5  s.color = GRAY
6  s.d = 0
7  s.π = NIL
8  Q = ∅
9  ENQUEUE(Q, s)
10 while Q ≠ ∅
11     u = DEQUEUE(Q)
12     for each v ∈ G.Adj[u]
13         if v.color == WHITE
14             v.color = GRAY
15             v.d = u.d + 1
16             v.π = u
17             ENQUEUE(Q, v)
18     u.color = BLACK
```
* connected components: repeated BFS
* bipartiteness: odd layer red, even layer blue. check color match

Complexity: $O(|V|+|E|)$.

TOPOLOGICAL-SORT(G)
```
1 DFS(G)
2 as each vertex is finished, insert it
      onto the front of a linked list
3 return the linked list of vertices
```
* if $v<w$, $v.f > w.f$.
DAG.

Complexity: $O(|V|+|E|)$

DFS(G)
```
1  for each vertex u ∈ G.V
2      u.color = WHITE
3      u.π = NIL
4  time = 0
5  for each vertex u ∈ G.V
6      if u.color == WHITE
7          DFS-VISIT(G, u)
```
* 8-queens * maze.

DFS-VISIT(G,u)
```
1  time = time + 1
2  u.d = time
3  u.color = GRAY
4  for each v ∈ G.Adj[u]
5      if v.color == WHITE
6          v.π = u
7          DFS-VISIT(G, v)
8  u.color = BLACK
9  time = time + 1
10 u.f = time
```

Complexity: $O(|V|+|E|)$.
Nesting Property: if $u \supset v$. $[u.d, u.f] \subset [v.d, v.f]$

KASARAJU'S-SCC(G)
```
1 DFS(G)
2 compute G^T
3 DFS(G^T), but in the main loop
      in order of decreasing u.f
4 output every DFS tree
```

Completxity: $O(|V|+|E|)$

MST-PRIM(G,w) unique path. $n-1$ edges.
```
1  H = {an arbitrary node v ∈ G.V}
2  F = ∅, R = ∅
3  for each vertex u ∈ G.Adj[v]
4      R = R ∪ {(u, v)}
5  while H ≠ G.V
6      e = EXTRACT-MIN(R)
7      let e = (u, v), with v ∉ H
8      H = H ∪ {v}
9      F = F ∪ {e}
10     for each vertex u ∈ G.Adj[v]
11         if u ∉ H
12             R = R ∪ {(u, v)}
```
priority queue

Complexity: $O(|E| \log |V|)$

MST-KRUSKAL(G,w)
```
1  A = ∅
2  for each vertex v ∈ G.V
3      MAKE-SET(v)
4  sort G.E in nondecreasing order of e.w
5  for each edge (u, v) ∈ G.E
6      if FIND-SET(u) ≠ FIND-SET(v)
7          A = A ∪ {(u, v)}
8          UNION(u, v)
9  return A
```

Complexity: $O(|E| \log |V|)$

INITIALIZE-SINGLE-SOURCE(G,s)
```
1 for each vertex v ∈ G.V
2     v.d = ∞
3     v.π = NIL
4 s.d = 0
```
* triangle inequality: $\delta(s,v) \le \delta(s,u) + w(u,v)$

RELAX(u,v,w)
```
1 if v.d > u.d + w(u, v)
2     v.d = u.d + w(u, v)
3     v.π = u
```
* shortest subpaths.

BELLMAN-FORD(G,w,s)
```
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  for i = 1 to |G.V| − 1
3      for each edge (u, v) ∈ G.E
4          RELAX(u, v, w)
5  for each edge (u, v) ∈ G.E
6      if v.d > u.d + w(u, v)
7          return FALSE
8  return TRUE
```

Complexity: $O(|V||E|)$

DIJKSTRA(G,w,s)
```
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  S = ∅
3  Q = G.V
4  while Q ≠ ∅
5      u = EXTRACT-MIN(Q)
6      S = S ∪ {u}
7      for each vertex v ∈ G.Adj[u]
8          RELAX(u, v, w)
```

Complexity: $O(|E| \log |V|)$ using a heap
$O(|V| \log |V| + |E|)$ using a Fibonacci heap

EXTEND-SHORTEST-PATHS(L,W)
```
1  n = L.rows
2  let L' = (l'_ij) be a new n × n matrix
3  for i = 1 to n
4      for j = 1 to n
5          l'_ij = ∞
6          for k = 1 to n
7              l'_ij = min(l'_ij, l_ik + w_kj)
8  return L'
```
repeated doubling.

Complexity: $O(n^3 \log n)$ $n \neq 2^k \Rightarrow n = \sum a_i 2^i$ (binary)

FLOYD-WARSHALL(W)
```
1  n = W.rows
2  D^(0) = W
3  for k = 1 to n
4      let D^(k) = (d^(k)_ij)
           be a new n × n matrix
5      for i = 1 to n
6          for j = 1 to n
7              d^(k)_ij = min(d^(k−1)_ij,
                   d^(k−1)_ik + d^(k−1)_kj)
8  return D^(n)
```

Complexity: $O(n^3)$

* 2-SAT:
if $x \rightsquigarrow \neg x$, $\neg x \rightsquigarrow x$. unsatisfiable.
assign value to reachable nodes.
repeat for remaining nodes

* MST principle: M partial MST.
$C \subset M$. e = min weight edge between Cs. $\Rightarrow M \cup \{e\}$ partial MST.

* Arborescenes: $w^*(u,v) = w(u,v) - \delta(v)$
* Johnson's: $h(v) = \delta(u,v)$ reweighing
$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$ ⇓ 非负.
$d_{uv} = \hat{\delta}(u,v) + h(v) - h(u)$

DP weighted interval scheduling

- intervals ordered with finishing time ↑. $I_1, I_2, \dots, I_n$
- $p(j) = \max\{k: I_k$ finishes before I_j starts$\}$. $/=0$ if no such I_k. $(j \geq 1)$
- $OPT(j) = \max\{v_j + OPT(p(j)), OPT(j-1)\}$
- 一维 array 从小到大填. $OPT(0)=0$
- $O(n)$ time if $p(j)$ known / $O(n \log n)$

subset sum / 0/1 knapsack

- $OPT(i, w') = \max\{OPT(i-1, w')$ $(v_i) w_i + OPT(i-1, w'-w_i)\}$
- 二维 $n \times W$ table M
  $M[0, w'] = 0$ for all $w' \leq W$
- for $i=1$ to n
  for $w=0$ to W
  ... // $O(nW)$ {space / time

complete knapsack 完全背包

- $OPT(i, w') = \begin{cases} OPT(i-1, w'), & w' < w_i \\ \max\{OPT(i-1, w'), OPT(i, w'-w_i)+v_i\}, & w' \geq w_i \end{cases}$
- for $i=1$ to n // $O(nW)$ {space / time
  for $w=0$ to w_i
  $M[i, w] = M[i-1, w]$
  for $w=w_{i+1}$ to W
  $M[i, w] = \max\{\dots\}$

longest common subsequence (LCS)

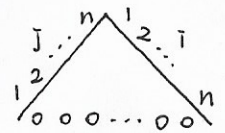
- $c(i, j) = len(LCS(X[1..i], Y[1..j]))$
- 二维 $n \times m$ table C
  $c(i, 0) = 0$, $c(0, j) = 0$ for $\forall i, j$.
- $c(i, j) = \begin{cases} c(i-1, j-1)+1, & X[i] = Y[j] \\ \max(c(i-1, j), c(i, j-1)), & X[i] \neq Y[j] \end{cases}$
- for $i=1$ to m
  for $j=1$ to n
  ...
- $O(nm)$ time/space
- making arrows to whichever cell $c(i,j)$ comes from, diagonal arrows indicating matching letters.

edit distance

- 二维 $n \times m$ table E
- $E[i,0]=i$, $E[0,j]=j$, for $\forall i, j$.
- $E[i,j] = \begin{cases} E[i-1, j-1], & X[i] = Y[j] \\ \min\{E[i-1, j-1]+1 \text{ (替换)}, \\ \quad E[i-1, j]+1, E[i, j-1]+1 \text{ (删)}\} \end{cases}$
- 同上. $O(nm)$ time/space
- making arrows.
  ↑/←: 删. ↖: match/替换

matrix multiplication

- $M[i,j]$: $A_i A_{i+1} \cdots A_j$ 最短时间
- $M[i,j] = \min_{i \leq k < j}\{M[i,k] + M[k+1, j] + p_{i-1} p_k p_j\}$
- $M[i,i] = 0$. $M[i, i+1] = p_{i-1} p_i p_{i+1}$
- 二维. 半个 $n \times n$ table M.



从下往上一层一层算 ($j-i$ 从小到大)

- another similar table to record where to break.
- $O(n^2)$ space. $O(n^3)$ time

common elements

{optimal substructure
overlapping subproblems
(small number: polynomial)

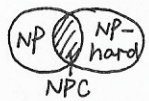
shortest path √ (relaxation)
longest simple path × 子问题不独立

longest palindrome sequence (LPS)

- $p[i,j] = len(LPS(S[i..j]))$ / $=-1$ if $j<i$.
- $last(s, T)$: the index of the last occurrence of s in T.
- $p[i,i]=1$, $\forall i>0$.
- $p[i,j] = \max\{p[i+1, j], (S_i \notin LPS)$
  $2 + p[i+1, last(S_i, S[i..j])-1]\}$ $(S_i \in LPS)$
- 二维 $n \times n$ table p.
- $O(n^2)$ time/space

树形DP状态转移至子树
根据给定复杂度猜子问题/分割

NP/NPC



NP-hard: $B \leq_P A$ for $\forall B \in NP$.

NP: solvability of an instance can be verified in poly time (with a certificate)

NP 证明

poly-time verifier $V(x,y)$
x: instance. y: certificate
$V(x,y) \in \{0,1\}$
x yes instance $\Leftrightarrow \exists y: V(x,y)=1$
x no instance $\Leftrightarrow \forall y: V(x,y)=0$

证明格式:

Verifier
Certificate y is _____.
Check that (要确定的性质)
If so, output 1, else output 0.

If x has a solution
then (有这么一个解).
Call it y and give y to V.
Clearly V outputs 1.

If x has no solution
then (不存在...).
So V outputs 0, no matter what y it gets.

V runs in poly-time.
... □

NPC 证明

$A \leq_P B$: $\exists f: A \to B$ s.t. for $\forall x$ of A, x is a yes instance of $A \Leftrightarrow f(x)$ is a yes instance of B.

$A \leq_P B$: poly-time reduction.
传递性.

证明格式:

reduction
give a mapping f. (指定B参数x)
Show it runs in poly time.

correctness
$\begin{cases} x \in A \Rightarrow f(x) \in B. \\ y \in B \Rightarrow f^{-1}(y) \in A. \end{cases}$ □

Eg. reduction

3-CNF-SAT $\leq_P$ CLIQUE

ϕ: 3-CNF with m clauses.
$\Rightarrow 3m$ vertices in G.
add an edge (u,v) if both
$\begin{cases} u, v$ from different clauses
$u \neq \neg v$ and $v \neq \neg u \end{cases}$
m-CLIQUE.

3-CNF-SAT $\leq_P$ SUBSET-SUM

ϕ: 3-CNF with n variables, k clauses.
S: $\{2n+2k$ numbers in base 10$\}$
two for each variable / clause
$n+k$ digits
one for each variable/clause
variable numbers v_i & v_i'
$v_i = v_i' = 1$ in x_i's digit
$v_i = 1$ in C_j's digit if $x_i \in C_j$
$v_i' = 1$ in C_j's digit if $\neg x_i \in C_j$
otherwise 0.
clause numbers s_j & s_j'
$s_j = 1$ in C_j's digit
$s_j' = 2$ in C_j's digit
otherwise 0.
target t
1 in variable digits
4 in clause digits

CLIQUE $\leq_P$ MATRIX-FIXING

$M_{u,v} \begin{cases} 1, & u=v \text{ or } \exists (u,v) \\ 0 & \text{otherwise} \end{cases}$
k-clique $\Leftrightarrow (n-k)$ operations

SAT 衍生 考虑加变量.

lower bound

indistinguishability

too few operations.
deliberately give an instance with wrong answer.

amortized analysis

average over worst case
potential function $\Phi: D \to R$ (状态 $\Rightarrow$ 值)
actual cost c_i
amortized cost
$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$
强制 $\Phi(D_0)=0$, $\Phi(D_i) \geq 0$ for $\forall i$.
$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0) \geq \sum_{i=1}^{n} c_i$

Eg. binary counter

$\Phi(D_i)$ = number of 1's in D_i
调整前面系数.
使 expensive operations Φ 下降多
normal operations Φ 升一点点

Fibonacci heap (min heap property)

make-heap: $O(1)$
insert: add to root list. $O(1)$
change H.min if necessary
find-min: H.min. $O(1)$
union: 合并 root lists. $O(1)$
H.min = min(H1.min, H2.min)
extract-min: remove H.min.
consolidate. $O(\log n)$

把 H.min 的孩子加到 root list
更新 H.min.
相同 degree (孩子数) root key
大的变成小的 root 的孩子.
直到没有相同 degree.

decrease-key: O(1)
violating min heap property.
⇒ cut out (移到 root list)
mark: 一个孩子被 cut out
第二个孩子被 cut out 则本身
也被 cut out
cascading cut 直到 root.

delete: decrease-key → −∞.
extract-min. O(logn)

String matching.

Brute-force: O(mn)

Knuth-Morris-Pratt (KMP).
failure function f(j): len
of longest matching proper
prefix & suffix of S[1..j]

```
function FF(S, m)   O(m)
  i←2, j←1
  f(1)←0
  while (i≤m)
    if S[i]=S[j]
      f(i)←j
      i←i+1, j←j+1
    else if j>1
      j←f(j-1)+1
    else
      f(i)←0
      i←i+1
  return f
```

```
function KMP(S,T)  O(n)
  m←len(S), n←len(T)
  i←1, j←1
  while (i≤n) and (j≤m)
    if T[i]=S[j]
      i←i+1, j←j+1
    else if j=1
      i←i+1
    else
      j←f[j-1]+1
  if j=m+1
    return i-m
  return "S not in T"
```

Rabin-Karp
O(m+n) most(average) case
O(mn) worst case

```
function Rabin-Karp(S,T,d,q)
  n←len(T), m←len(S)
  h←d^(m-1) mod q
  p←0, t0←0
  for i←1 to m
    p←(d·p+S[i]) mod q
    t0←(d·t0+T[i]) mod q
  for s←0 to n-m
    if p=ts
      if S[1..m]=T[s+1..s+m]
        print s
    if s<n-m
      ts+1=(d(ts-T[s+1]·h)+T[s+m+1]) mod q
```

CS140 Midterm Review

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1 Mathematical Background

1.1 Summations

1.1.1 Sums of squares and cubes

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

1.1.2 Harmonic series

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$$

1.2 Stirling's approximation

$$\log n! = n \log n - (\log e)n + O(\log n)$$

2 Analysis of algorithms

2.1 Master theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n)$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Note: In cases 1 (case 3), $f(n)$ needs to be smaller (larger) than $n^{\log_b a}$ by a **polynomial factor** n^{ϵ}, so we can't always apply Master theorem.

E.g. $T(n) = 2T(n/2) + n \log n$, $\log_b a = 1$, $f(n) \neq \Theta(n)$ and $f(n) \neq \Omega(n^{1+\epsilon})$, can't use Master theorem to solve this recurrence.

2.2 Big O analysis

2.2.1 Asymptotic notation

notation	set of functions form	limit form
$O(g(n))$	$\{f(n) : \exists\ c > 0,\ n_0 > 0,\ \forall\ n \geq n_0,\ 0 \leq f(n) \leq cg(n)\}$	$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} < \infty$
$\Omega(g(n))$	$\{f(n) : \exists\ c > 0,\ n_0 > 0,\ \forall\ n \geq n_0,\ 0 \leq cg(n) \leq f(n)\}$	$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} > 0$
$\Theta(g(n))$	$\{f(n) : \exists\ c_1 > 0,\ c_2 > 0,\ n_0 > 0,\ \forall\ n \geq n_0,\ 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$	$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = c,\ 0 < c < \infty$
$o(g(n))$	$\{f(n) : \forall\ c > 0,\ \exists\ n_0 > 0,\ \forall\ n \geq n_0,\ 0 \leq f(n) < cg(n)\}$	$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$
$\omega(g(n))$	$\{f(n) : \forall\ c > 0,\ \exists\ n_0 > 0,\ \forall\ n \geq n_0,\ 0 \leq cg(n) < f(n)\}$	$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = \infty$

2.2.2 Big O properties

1. $f(n) = O(g(n)),\ f(n) = \Omega(g(n)) \implies f(n) = \Theta(g(n))$
2. $O,\ \Omega,\ \Theta$ are **transitive**.
3. Θ is **symmetric**.
4. $O(1)$ is the set of constants.

3 Sorting

3.1 Comparison sorts

$\Omega(n \log n)$ lower bound on time complexity

3.1.1 Insertion sort

INSERTION-SORT(A)

```
1  for j = 2 to A.length
2      key = A[j]
3      // Insert A[j] into the sorted sequence A[1..j-1].
4      i = j - 1
5      while i > 0 and A[i] > key
6          A[i+1] = A[i]
7          i = i - 1
8      A[i+1] = key
```

Complexity: $O(n^2)$
Pros:
Simple algorithm with little bookkeeping overhead.
Requires only one unit of extra storage, for swap.
Fast if input is nearly sorted.
Effective for small n (e.g. $n \leq 20$), due to low overhead, and $O(n^2)$ being not too big for small n.
Cons: not practical for large input size because $O(n^2)$ is too high

3.1.2 Mergesort

MERGE(A, p, q, r)

```
 1  n_1 = q - p + 1
 2  n_2 = r - q
 3  let L[1..n_1 + 1] and R[1..n_2 + 1] be new arrays
 4  for i = 1 to n_1
 5      L[i] = A[p + i - 1]
 6  for j = 1 to n_2
 7      R[j] = A[q + j]
 8  L[n_1 + 1] = ∞
 9  R[n_2 + 1] = ∞
10  i = 1
11  j = 1
12  for k = p to r
13      if L[i] ≤ R[j]
14          A[k] = L[i]
15          i = i + 1
16      else
17          A[k] = R[j]
18          j = j + 1
```

MERGE-SORT(A, p, r)

```
1  if p < r
2      q = ⌊(p + r)/2⌋
3      MERGE-SORT(A, p, q)
4      MERGE-SORT(A, q + 1, r)
5      MERGE(A, p, q, r)
```

Complexity: $O(n \log n)$
Pros: much faster for large n; can be parallelized
Cons: needs extra storage in L, R, $2n$ storage for sorting n numbers
Combining mergesort and insertion sort:
Divide problem into small groups (≈ 20 elements each).
Use insertion sort to sort each group.
Combine the sorted groups using merge.

3.1.3 Quicksort

PARTITION(A, p, r)

```
1  x = A[r]
2  i = p - 1
3  for j = p to r - 1
4      if A[j] ≤ x
5          i = i + 1
6          exchange A[i] with A[j]
7  exchange A[i + 1] with A[r]
8  return i + 1
```

Complexity: $O(n)$

QUICKSORT(A, p, r)

```
1  if p < r
2      q = PARTITION(A, p, r)
3      QUICKSORT(A, p, q − 1)
4      QUICKSORT(A, q + 1, r)
```

Complexity: depends on how **balanced** the pivot is.

Best case: $u = v$, $O(n \log n)$

Worst case: $u = n - 1$, $v = 0$, $O(n^2)$

Average case: $O(n \log n)$ (use **recursion tree** to see if $u = 0.9n$, $v = 0.1n - 1$)

Pros: generally a little faster than mergesort; needs no additional storage; can be parallelized

Comparing mergesort and quicksort:

In mergesort, the division is trivial, but combination step takes $O(n)$ time.

Quicksort spends more time finding a nice division of the problem, to make the combination step faster.

Overall time complexity of mergesort and Quicksort are the same (assuming random input).

3.2 Sorting in linear time

3.2.1 Counting sort

Assumes all input values are integers in the range 0 to k, for some k. Used as a stand-alone algorithm, and also as a subroutine in other algorithms, e.g., radix sort, which ensures k is small when counting sort is fast.

COUNTING-SORT(A, B, k)

```
 1  let C[0..k] be a new array
 2  for i = 0 to k
 3      C[i] = 0
 4  for j = 1 to A.length
 5      C[A[j]] = C[A[j]] + 1
 6  // C[i] now contains the number of elements equal to i.
 7  for i = 1 to k
 8      C[i] = C[i] + C[i − 1]
 9  // C[i] now contains the number of elements less than or equal to i.
10  for j = A.length downto 1
11      B[C[A[j]]] = A[j]
12      C[A[j]] = C[A[j]] − 1
```

Complexity: $\Theta(n + k)$. If $k = O(n)$, $\Theta(n)$.

Pros: stable

Cons: becomes increasingly inefficient as k gets larger

3.2.2 Radix sort

RADIX-SORT(A, d)

```
1  for i = 1 to d
2      use a stable sort to sort array A on digit i
```

Complexity:

1. n d-digit numbers, where each digit is between 0 and $k - 1$, $\Theta(d(n + k))$

2. n b-bit numbers, break each number into blocks of $r \leq b$ bits, $\Theta(\frac{b}{r}(n+2^r))$

3. setting $r = \min(\lfloor \log n \rfloor, b)$ minimizes the running time $\Theta(\frac{b}{r}(n+2^r))$

3.2.3 Bucket sort

Works for real value inputs in $[0, 1)$. Can shift and scale inputs so they always fall in this range.

BUCKET-SORT(A)
1 let $B[0..n-1]$ be a new array
2 $n = A.length$
3 **for** $i = 0$ **to** $n-1$
4 make $B[i]$ an empty list
5 **for** $i = 1$ **to** n
6 insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
7 **for** $i = 0$ **to** $n-1$
8 sort list $B[i]$ with insertion sort
9 concatenate the lists $B[0]$, $B[1]$, ..., $B[n-1]$ together in order

Complexity: $\Theta(n)$ if inputs are **randomly distributed**.

Proof. Total running time $T(n)$ equals $\Theta(n)$ time distributing input values to buckets plus $\sum_{i=0}^{n-1} O(n_i^2)$ time sorting all the buckets using insertion sort.

$$\begin{aligned}
\mathrm{E}[T(n)] &= \mathrm{E}\left[\Theta(n) + \sum_{i=0}^{n-1} O(n_i^2)\right] \\
&= \Theta(n) + \sum_{i=0}^{n-1} \mathrm{E}[O(n_i^2)] \\
&= \Theta(n) + \sum_{i=0}^{n-1} O(\mathrm{E}[n_i^2])
\end{aligned}$$

Let $X_{ij} = 1$ if the jth input value falls into the ith bucket, 0 otherwise. $n_i = \sum_{j=1}^{n} X_{ij}$ is the number of inputs in the ith bucket.

$$\begin{aligned}
\mathrm{E}[n_i^2] &= \mathrm{E}\left[\left(\sum_{j=1}^{n} X_{ij}\right)^2\right] \\
&= \mathrm{E}\left[\sum_{j=1}^{n}\sum_{k=1}^{n} X_{ij}X_{ik}\right] \\
&= \mathrm{E}\left[\sum_{j=1}^{n} X_{ij}^2 + \sum_{1\leq j\leq n}\sum_{\substack{1\leq k\leq n \\ k\neq j}} X_{ij}X_{ik}\right] \\
&= \sum_{j=1}^{n} \mathrm{E}[X_{ij}^2] + \sum_{1\leq j\leq n}\sum_{\substack{1\leq k\leq n \\ k\neq j}} \mathrm{E}[X_{ij}X_{ik}]
\end{aligned}$$

As $\mathrm{E}[X_{ij}^2] = 1^2 \cdot \frac{1}{n} + 0^2 \cdot \left(1 - \frac{1}{n}\right) = \frac{1}{n}$, $\mathrm{E}[X_{ij}X_{ik}] = \mathrm{E}[X_{ij}]\mathrm{E}[X_{ik}] = \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$,

$$\begin{aligned}
\mathrm{E}[n_i^2] &= \sum_{j=1}^{n} \frac{1}{n} + \sum_{1 \le j \le n} \sum_{\substack{1 \le k \le n \\ k \ne j}} \frac{1}{n^2} \\
&= n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n^2} \\
&= 1 + \frac{n-1}{n} \\
&= 2 - \frac{1}{n}
\end{aligned}$$

Thus $\mathrm{E}[T(n)] = \Theta(n) + \sum_{i=0}^{n-1} O(\mathrm{E}[n_i^2]) = \Theta(n) + \sum_{i=0}^{n-1} O\left(2 - \frac{1}{n}\right) = \Theta(n)$. □

3.3 Heapsort

3.3.1 Heap

Depth of a node: the length of the path from the node to the root

Height of a tree: the largest depth of any node

Complete binary trees: all layers, except possibly the last, are full; nodes in the bottom layer are as far left as possible

height $= h \iff$ # of nodes $\in [2^h, 2^{h+1} - 1]$

# of nodes $= n \iff$ height $= \lfloor \log_2 n \rfloor$

Heap: a complete binary tree where each node contains a value; the value of any node is larger (smaller) than the value of any other node in its **subtree** in a max (min) heap.

3.3.2 Heapify

```
LEFT(i)
1  return 2i

RIGHT(i)
1  return 2i + 1

MAX-HEAPIFY(A, i)
 1  l = LEFT(i)
 2  r = RIGHT(i)
 3  if l ≤ A.heap-size and A[l] > A[i]
 4       largest = l
 5  else
 6       largest = i
 7  if r ≤ A.heap-size and A[r] > A[largest]
 8       largest = r
 9  if largest ≠ i
10       exchange A[i] with A[largest]
11       MAX-HEAPIFY(A, largest)
```

Complexity: $O(h - d)$ where h is the height of the heap and d is the depth of the node

3.3.3 Building a heap

Label nodes left to right, top to bottom, heapify **internal nodes** in **reverse label order** (right to left, bottom up).

```
BUILD-MAX-HEAP(A)
1  A.heap-size = A.length
2  for i = ⌊A.length/2⌋ downto 1
3      MAX-HEAPIFY(A, i)
```

Complexity: $O(n)$

3.3.4 Heapsort

```
HEAPSORT(A)
1  BUILD-MAX-HEAP(A)
2  for i = A.length downto 2
3      exchange A[1] with A[i]
4      A.heap-size = A.heap-size − 1
5      MAX-HEAPIFY(A, 1)
```

Complexity: $O(n \log n)$
Cons: generally slower than others due to cache effects; cannot be parallelized

4 Search

4.1 Binary search

```
BINARY-SEARCH(A, x, l, r)
1  m = ⌊(l + r)/2⌋
2  if x == A[m]
3      return m
4  if l == r
5      return NIL
6  if x < A[m]
7      return BINARY-SEARCH(A, x, l, m − 1)
8  return BINARY-SEARCH(A, x, m + 1, r)
```

Complexity: $O(\log n)$

4.2 Ternary search

A function f is unimodal if there is an x s.t. f is monotonically decreasing for $y \leq x$, and monotonically increasing for $y > x$. Thus f has a unique minimum at x. Can also consider symmetric case where f first increases then decreases. In this case f has a unique maximum. Our goal is to find x.

TERNARY-SEARCH(f, l, r, ϵ)

```
1  if r − l < ε
2        return (r + l)/2
3  x = l + (r − l)/3
4  y = r − (r − l)/3
5  if f(x) < f(y)
6        return TERNARY-SEARCH(f, l, y, ε)
7  if f(x) > f(y)
8        return TERNARY-SEARCH(f, x, r, ε)
9  return TERNARY-SEARCH(f, x, y, ε)
```

Complexity: $O(\log(\frac{r-l}{\epsilon}))$

4.3 Selection using median of medians

MEDIAN-OF-MEDIANS(A, p, r)

```
1  l = ⌈(r − p + 1)/5⌉
2  let M[1 . . l] be a new array
3  for k = 1 to l − 1
4        M[k] = A[SELECT(A, 5k + p − 5, 5k + p − 1, 3)]
5  M[l] = A[SELECT(A, 5l + p − 5, r, ⌈(r − 5l − p + 6)/2⌉)]
6  m = SELECT(M, 1, l, ⌊l/2⌋)
7  if m == l
8        return ⌊(5l + p − 5 + r)/2⌋
9  return 5m + p − 3
```

SELECT(A, p, r, i)

```
 1  if r − p + 1 ≤ 5
 2        INSERTION-SORT(A[p . . r])
 3        return p − 1 + i
 4  exchange A[r] with A[MEDIAN-OF-MEDIANS(A, p, r)]
 5  q = PARTITION(A, p, r)
 6  k = q − p + 1
 7  if i == k
 8        return q
 9  elseif i < k
10        return SELECT(A, p, q − 1, i)
11  else
12        return SELECT(A, q + 1, r, i − k)
```

Complexity: $S(n) = S(n/5) + S(u) + O(n)$ where u is the size of whichever partition we recurse on
$O(n)$ finding medians of $\lceil(n/5)\rceil$ groups of 5 numbers and partitioning the array
$S(n/5)$ finding the median of the $\lceil(n/5)\rceil$ medians
$u \leq 7n/10$, $S(n) = O(n)$, but the hidden constant term is **large**

4.4 Randomized selection

RANDOMIZED-PARTITION(A, p, r)

```
1  i = RANDOM(p, r)
2  exchange A[r] with A[i]
3  return PARTITION(A, p, r)
```

RANDOMIZED-SELECT(A, p, r, i)

```
1  if p == r
2        return A[p]
3  q = RANDOMIZED-PARTITION(A, p, r)
4  k = q − p + 1
5  if i == k        // the pivot value is the answer
6        return A[q]
7  elseif i < k
8        return RANDOMIZED-SELECT(A, p, q − 1, i)
9  else
10       return RANDOMIZED-SELECT(A, q + 1, r, i − k)
```

Complexity: expected linear time

5 Binary search trees

5.1 Operations

Complexity: all $O(h)$

5.1.1 Find

FIND(x, k)

```
1  if x == NIL or k == x.key
2        return x
3  if k < k.key
4        return FIND(x.left, k)
5  return FIND(x.right, k)
```

ITERATIVE-FIND(x, k)

```
1  while x ≠ NIL and k ≠ x.key
2        if k < x.key
3              x = x.left
4        else
5              x = x.right
6  return x
```

5.1.2 Insert

INSERT(T, k)

```
 1  x = T.root
 2  if FIND(x, k) == NIL
 3       allocate a new node z
 4       z.key = k
 5       z.left = NIL
 6       z.right = NIL
 7       y = NIL
 8       while x ≠ NIL
 9            y = x
10            if z.key < x.key
11                 x = x.left
12            else
13                 x = x.right
14       z.p = y
15       if y == NIL
16            T.root = z        // tree T was empty
17       elseif z.key < y.key
18            y.left = z
19       else
20            y.right = z
```

5.1.3 Successor

MINIMUM(x)

```
1  while x.left ≠ NIL
2       x = x.left
3  return x
```

SUCCESSOR(x)

```
1  if x.right ≠ NIL
2       return MINIMUM(x.right)
3  y = x.p
4  while y ≠ NIL and x == y.right
5       x = y
6       y = y.p
7  return y
```

5.1.4 Delete

TRANSPLANT(T, u, v)

```
1  if u.p == NIL
2       T.root = v
3  elseif u == u.p.left
4       u.p.left = v
5  else
6       u.p.right = v
7  if v ≠ NIL
8       v.p = u.p
```

```
DELETE(T, z)
 1  if z.left == NIL
 2      TRANSPLANT(T, z, z.right)
 3  elseif z.right == NIL
 4      TRANSPLANT(T, z, z.left)
 5  else
 6      y = MINIMUM(z.right)
 7      if y.p ≠ z
 8          TRANSPLANT(T, y, y.right)
 9          y.right = z.right
10          y.right.p = y
11      TRANSPLANT(T, z, y)
12      y.left = z.left
13      y.left.p = y
```

5.2 Rotations

```
LEFT-ROTATE(T, x)
 1  y = x.right              // set y
 2  x.right = y.left         // turn y's left subtree into x's right subtree
 3  if y.left ≠ T.nil
 4      y.left.p = x
 5  y.p = x.p                // link x's parent to y
 6  if x.p == T.nil
 7      T.root = y
 8  elseif x == x.p.left
 9      x.p.left = y
10  else
11      x.p.right = y
12  y.left = x               // put x on y's left
13  x.p = y
```

```
RIGHT-ROTATE(T, y)
 1  x = y.left               // set x
 2  y.left = x.right         // turn x's right subtree into y's left subtree
 3  if x.right ≠ T.nil
 4      x.right.p = y
 5  x.p = y.p                // link y's parent to x
 6  if y.p == T.nil
 7      T.root = x
 8  elseif y == y.p.left
 9      y.p.left = x
10  else
11      y.p.right = x
12  x.right = y              // put y on x's right
13  y.p = x
```

Complexity: $O(1)$

5.3 AVL trees

AVL invariant: $bal(x) = ht(x.r) - ht(x.l)$ at all nodes is -1, 0 or 1

Insert: BST-INSERT first, then find the **lowest unbalanced node** and use $O(1)$ (one or two) rotations to restore balance.

Delete: BST-DELETE first, then **bottom up** find unbalanced node and use $O(h)$ rotations to restore balance.

Complexity: $O(h) = O(\log n)$

6 Hashing

6.1 Direct addressing

- Suppose we want to store $(key, value)$ pairs, where keys come from a finite universe $U = \{0, 1, \ldots, m-1\}$
- Use an array of size m. All operations take $O(1)$ time.
  - INSERT(k, v) Store v in array position k.
  - FIND(k) Return the value in array position k.
  - DELETE(k) Clear the value in array position k.
- Cons:
  - If m is large, then we need to use a lot of memory. Uses $O(|U|)$ space.
  - If only need to store few values, lots of space wasted.

6.2 Hash tables

- A randomized data structure to efficiently implement a dictionary.
- Support find, insert, and delete operations all in expected $O(1)$ time, but in the worst case, all operations are $O(n)$. The worst case is provably very unlikely to occur.
- Do not support efficient min/max or predecessor/successor functions. All these take $O(n)$ time on average.
- A practical, efficient alternative to binary search trees if only find, insert and delete needed.
- Consists of the following:
  - A universe U of keys.
  - An array T of size m.
  - A hash function $h : U \to \{0, 1, \ldots, m-1\}$.
- Assuming $h(k)$ takes $O(1)$ time to compute, all operations still take $O(1)$ time. Uses $O(m)$ space, much less than direct addressing if $m \ll |U|$.
  - INSERT(k, v) Hash key to $h(k)$. Store v in $T[h(k)]$.
  - FIND(k) Return the value in $T[h(k)]$.
  - DELETE(k) Delete the value in $T[h(k)]$.
- Collisions unavoidable when $|U| > m$ by Pigeonhole Principle.

6.3 Closed addressing

- Every entry in hash table points to a linked list. Keys that hash to the same location get added to the linked list.
- Suppose the longest list has length $\hat{n}$, and average length list is $\bar{n}$. Each operation takes worst case $O(\hat{n})$ time. An operation on a random key takes $O(\bar{n})$ time.
  - INSERT(k) Add k to the linked list in $T[h(k)]$.
  - FIND(k) Search the linked list in $T[h(k)]$ for k.
  - DELETE(k) Delete k from the linked list in $T[h(k)]$.
- Suppose the hash table contains n items, and has size m. Call $\alpha = n/m$ **load factor**. The average time for each operation is $O(\alpha)$. However, even with uniform hashing, in the worst case, all keys can hash to the same location, so the worst case performance is still $O(n)$.

6.3.1 Heuristic hash functions

- Assume the keys are natural numbers. Convert other data types to numbers.
- **Division method:** $h(k) = k \bmod m$, often choose m a prime number not too close to a power of 2.
- **Multiplication method:** $h(k) = \lfloor m(kA \bmod 1) \rfloor$, where A is a constant in the range $(0, 1)$.

6.3.2 Universal hashing

- No matter what the n input keys are, every operation takes optimal $O(n/m)$ time in expectation, for a size m hash table.
- **Universal hash family** H: $\forall$ keys $x \neq y$, $P_{h \in H}[h(x) = h(y)] \leq 1/m$.
- Let S be a set of n keys, and let $x \in S$. If $h \in H$ is chosen at random, then the expected number of $y \in S$ such that $h(x) = h(y)$ is n/m.
- **Constructing a universal hash family**
  - Choose a prime number p such that $p > m$, and all keys $< p$.
  - $h_{ab}(k) = ((ak + b) \bmod p) \bmod m$
  - $H_{pm} = \{h_{ab} \mid a \in \{1, 2, \ldots, p-1\}, b \in \{0, 1, \ldots, p-1\}\}$

6.4 Open addressing

Define a new hash function $h : U \times \{0, 1, \ldots, m-1\} \to \{0, 1, \ldots, m-1\}$ and a **probe sequence** $\langle h(k, 0), h(k, 1), \ldots, h(k, m-1) \rangle$.

6.4.1 Operations

HASH-INSERT(T, k)

```
1  i = 0
2  repeat
3       j = h(k, i)
4       if T[j] == NIL or T[j] == DELETED
5            T[j] = k
6            return j
7       else
8            i = i + 1
9  until i == m
10 error "hash table overflow"
```

HASH-SEARCH(T, k)

```
1  i = 0
2  repeat
3       j = h(k, i)
4       if T[j] == k
5            return j
6       i = i + 1
7  until T[j] == NIL or i == m
8  return NIL
```

HASH-DELETE(T, k)

```
1  T[HASH-SEARCH(T, k)] = DELETED
```

Now search time depends both on current number of table entries, and also number of past deleted ones, so **closed addressing** is more common when keys need to be deleted. Since each table entry stores one key, a table of size m can store at most m keys. However, since each table entry stores just the key and no pointers, then for the same amount of storage, an open addressing hash table can store more entries than a closed addressing table. Thus, it has lower load factor.

6.4.2 Probe sequences

- **Two Properties**
  - $h(k, 0), h(k, 1), \ldots, h(k, m-1)$ covers all table entries $0, 1, \ldots, m-1$.
  - Given a random key k, $h(k, 0), h(k, 1), \ldots, h(k, m-1)$ is a random sequence in the $m!$ permutations of $\{0, \ldots, m-1\}$.
- **Techniques to compute probe sequences**
  - **Linear probing:** $h(k, i) = (h'(k) + i) \bmod m$, where h' is an ordinary hash function. $h'(k)$ determines the entire probe sequence, so theres only m different probe sequences. Poor performance due to **primary clustering**. An empty slot with i filled slots before it gets filled with probability $(i+1)/m$, so long runs of filled slots, that tend to get even longer.
  - **Quadratic probing:** $h(k, i) = (h'(k) + c_1 i + c_2 i^2) \bmod m$, where c_1, c_2 are constants. No primary clustering, but **secondary clustering**, where $h(k_1, 0) = h(k_2, 0)$ implies both probe sequences are equal.

- **Double hashing:** $h(k,i) = (h_1(k) + ih_2(k)) \bmod m$. $h_2(k)$ needs to be relatively prime to m to make sure entire table searched. One way to ensure this is make m a power of 2, and h_2 always odd. Can also make m prime, $m' = m - 1$, and set $h_1(k) = k \bmod m$, $h_2(k) = 1 + (k \bmod m')$. Since each distinct $(h_1(k), h_2(k))$ leads to distinct probe sequence, double hashing can produce $O(m^2)$ probe sequences. Performs quite well in practice.

6.5 Perfect hashing

- Ensures **no colisions** for a **fixed set of keys**.
- Allows FIND(k) and DELETE(k) in $O(1)$ time. Does not support INSERT(k).
- Uses two levels of universal hashing.
  - Use first layer hash function h to hash key to a location in T.
  - Each location j in T points to a hash table S_j with hash function h_j.
  - If n_j keys hash to location j, the size of s_j is $m_j = n_j^2$.
- Suppose we store n keys in a hash table of size $m = n^2$ using universal hashing, then there is a $< 1/2$ probability of collision. If collisions occur, pick another random hash function from the universal family. In expectation, do this twice before finding a hash function causing no collisions.
- Suppose we store n keys in a hash table of size $m = n$. Then the secondary hash tables use space $E[\sum_{j=0}^{m-1} n_j^2] \leq 2n$, where n_j is the number of keys hashing to location j. Overall the sapce use is $O(m + \sum_{j=0}^{m-1} n_j^2) = O(m)$.

6.6 Bloom filters

- Can implement a set. It only keeps track of which keys are present, not any values associated to keys.
- Supports insert and find operations. Doesn not support delete operations. Deletes can be done by storing a count of how many keys hashed to that location, and incline/decline the counts when inserting or deleting. But this uses more memory. Also, what if the counts overflow?
- Use less memory than hash tables or other ways of implementing sets.
- **Approximate:** can produce **false positives**, no **false negatives**. False positive probability

$$f = \left[1 - \left(1 - \frac{1}{m}\right)^{nk}\right]^k \approx \left(1 - e^{-\frac{nk}{m}}\right)^k$$

  where k is the number of hash functions, m is the size of the table and n is the number of keys inserted. False positive probability minimized by $k = \frac{m \ln 2}{n}$, which leads to $f = (1/2)^k \approx 0.6185^{\frac{m}{n}}$, where m/n is the average number of bits per item. So error rate decreases exponentially in space usage.
- **Neat trick** Given Bloom filters for sets S_1, S_2, we can create Bloom filter for $S_1 \cap S_2$ and $S_1 \cup S_2$ just by bitwise ANDing or ORing S_1 and S_2's filters.
- Consists of:
  - An array A of size m, initially all 0's.
  - k **independent** hash functions $h_1, \ldots, h_k : U \to \{1, \ldots, m\}$.

BLOOM-FILTER-INSERT(A, H, x)

```
1  for each h ∈ H
2      A[h(x)] = 1
```

BLOOM-FILTER-SEARCH(A, H, x)

```
1  for each h ∈ H
2      if A[h(x)] == 0
3          return FALSE
4  return TRUE
```

7 Divide and conquer

7.1 Multiplication

7.1.1 Gauss's method for multiplying complex numbers

$(a + bi)(c + di) = x + yi$, where $x = ac - bd$, $y = ad + bc = (a + b)(c + d) - ac - bd$
4 multiplications + 2 additions $\rightarrow$ 3 multiplications + 5 additions

7.1.2 Karatsuba multiplication

To multiply binary numbers a and b, split their bits in half, $a = 2^{n/2} \cdot a_1 + a_0$, $b = 2^{n/2} \cdot b_1 + b_0$, then $ab = 2^n \cdot a_1b_1 + 2^{n/2} \cdot (a_1b_0 + a_0b_1) + a_0b_0 = 2^n \cdot a_1b_1 + 2^{n/2} \cdot ((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0) + a_0b_0$.
Complexity: $S(n) = 3S(n/2) + O(n)$, $S(n) = \Theta(n^{\log 3}) = O(n^{1.59})$

7.1.3 Strassen's algorithm for block matrix multiplication

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$P_1 = A_{11} \times (B_{12} - B_{22}) \qquad P_2 = (A_{11} + A_{12}) \times B_{22} \qquad P_3 = (A_{21} + A_{22}) \times B_{11}$$

$$P_4 = A_{22} \times (B_{21} - B_{11}) \qquad P_5 = (A_{11} + A_{22}) \times (B_{11} + B_{22})$$

$$P_6 = (A_{12} - A_{22}) \times (B_{21} + B_{22}) \qquad P_7 = (A_{11} - A_{21}) \times (B_{11} + B_{12})$$

$$C_{11} = P_5 + P_4 - P_2 + P_6 \qquad C_{12} = P_1 + P_2 \qquad C_{21} = P_3 + P_4 \qquad C_{22} = P_5 + P_1 - P_3 - P_7$$

Complexity: $S(n) = 7S(n/2) + O(n^2)$, $S(n) = \Theta(n^{\log 7}) = O(n^{2.81})$

7.1.4 FFT for polynomial multiplication

Details see Recitation2.pdf
Complexity: $O(n \log n)$

7.2 Counting inversions

COUNT-CROSSING-INVERSIONS(A, p, q, r)

```
1  n_1 = q − p + 1
2  n_2 = r − q
3  let L[0..n_1 − 1] and R[0..n_2 − 1] be new arrays
4  num = 0
5  for i = 0 to n_1 − 1
6       L[i] = A[p + i]
7  for j = 0 to n_2 − 1
8       R[j] = A[q + 1 + j]
9  i = 0
10 j = 0
11 k = p
12 while i < n_1 and j < n_2
13      if L[i] > R[j]
14           num = num + 1
15           A[k] = R[j]
16           j = j + 1
17      else
18           num = num + j
19           A[k] = L[i]
20           i = i + 1
21      k = k + 1
22 if j == n_2
23      num = num + (n_1 − i − 1) * n_2
24      for r = 0 to n_1 − i − 1
25           A[k + r] = L[i + r]
26 else
27      num = num − j
28      for r = 0 to n_2 − j − 1
29           A[k + r] = R[j + r]
30 return num
```

COUNT-INVERSIONS(A, p, r)

```
1  if p == r
2       return 0
3  if r == p + 1
4       if A[p] > A[r]
5            return 1
6       else
7            return 0
8  q = ⌊(p + r/2)⌋
9  return COUNT-INVERSIONS(A, p, q) + COUNT-INVERSIONS(A, q + 1, r)+
       COUNT-CROSSING-INVERSIONS(A, p, q, r)
```

Complexity: $T(n) = 2T(n/2) + O(n)$, $T(n) = O(n \log n)$

7.3 Maximum subarray

FIND-MAX-CROSSING-SUBARRAY$(A, low, mid, high)$

```
1  left-sum = −∞
2  sum = 0
3  for i = mid downto low
4       sum = sum + A[i]
5       if sum > left-sum
6            left-sum = sum
7            max-left = i
8  for j = mid + 1 to high
9       sum = sum + A[j]
10      if sum > right-sum
11           right-sum = sum
12           max-right = j
13 return (max-left, max-right, left-sum + right-sum)
```

FIND-MAXIMUM-SUBARRAY$(A, low, right)$

```
1  if high == low
2       return (low, high, A[low])
3  else
4       mid = ⌊(low + high)/2⌋
5       (left-low, left-high, left-sum) = FIND-MAXIMUM-SUBARRAY(A, low, mid)
6       (right-low, right-high, right-sum) = FIND-MAXIMUM-SUBARRAY(A, mid + 1, high)
7       (cross-low, cross-high, cross-sum) = FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
8       if left-sum ≥ right-sum and left-sum ≥ cross-sum
9            return (left-low, left-high, left-sum)
10      elseif right-sum ≥ left-sum and right-sum ≥ cross-sum
11           return (right-low, right-high, right-sum)
12      return (cross-low, cross-high, cross-sum)
```

7.4 Closest point pair

8 Greedy algorithms

8.1 Interval scheduling

8.2 Interval coloring

8.3 Huffman coding

HUFFMAN(C)

```
1  n = |C|
2  Q = C
3  for i = 1 to n − 1
4       allocate a new node z
5       z.left = x = EXTRACT-MIN(Q)
6       z.right = y = EXTRACT-MIN(Q)
7       z.freq = x.freq + y.freq
8       INSERT(Q, z)
9  return EXTRACT-MIN(Q)        // return the root of the tree
```

9 Graph algorithms

9.1 Graphs

9.1.1 Basic definitions

- $G = (V, E)$, where V is the set of **vertices (nodes)**, $E \subseteq V \times V$ is the set of **edges**.

$$|E| = O(|V|^2)$$

- An edge $e = (u, v)$ consists of two **endpoints**, u and v are **adjacent (neighbors)**. Two edges are **adjacent** if they share an endpoint.
- A **path** is a set of adjacent edges. A graph is **connected** if theres a path between any two nodes. If a graph is not connected, it consists of a set of **connected components**. **Distance** between two nodes is the smallest number of edges between the nodes in any path.
- A **cycle** is a path that starts and ends at the same vertex. A graph is **acyclic** if it doesnt have any cycles.
- **Degree** of a vertex v, $d(v)$, is the number of neighbors v has.
- **Handshaking theorem:** Let $G = (V, E)$ be an undirected graph, $\sum_{v \in V} d(v) = 2|E|$.

9.1.2 Types of graphs

- **directed** (follow edge in direction indicated), **undirected** (follow edge in either direction)
- **simple** (no parallel edges between nodes, no self loops), **cyclic** (following edges from a node, can get back to the same node)
- **explicit** (edges are listed), **implicit** (edges generated on the fly, e.g. in search trees)
- **weighted** (edges have values attached), **unweighted** (all edges have weight 1)
- **dense** (many edges in graph), **sparse** (O(n) edges in graph with n nodes)
- **embedded** (vertices have coordinates, e.g., in Euclidean space), **topological** (edges only indicate connections)
- **labeled** (nodes have labels (names) attached)

9.1.3 Trees

- A **tree** is a connected, acyclic undirected graph.
- A tree with n vertices has $n - 1$ edges.
- Removing any edge from a tree **disconnects** it into two **subtrees**.
- Adding any edge to a tree creates a **cycle**.
- There is a **unique path** between any two nodes in a tree.
- A **forest** is a collection of trees.

9.2 Representations of graphs

9.2.1 Adjacency list

An array of linked lists. Each array location represents a vertex. Linked list at a location gives (directed) neighbors of vertex. Size is $O(|E|)$, i.e., proportional to number of edges. Uses less memory if G doesnt have many edges, but to see if (i, j) is an edge, need to scan i's list in $O(|V|)$ time.

9.2.2 Adjacency matrix

$|V|$ rows and $|V|$ columns. Entry $(i, j) = 1$ if i is adjacent to j in an undirected graph and if there is an edge from i to j in a directed graph, 0 otherwise. Size is $O(|V|^2)$. Uses more memory, but can check if (i, j) is an edge in $O(1)$ time.

9.3 BFS

Use a queue Q storing nodes. Each node v has three fields:
$v.d$ giving v's distance to s
$v.color \in \{\text{WHITE}, \text{GRAY}, \text{BLACK}\}$ showing whether v is **undiscovered**, **being processed** or **processed**
$v.\pi$ giving v's parent

```
BFS(G, s)
 1  for each vertex u ∈ G.V − {s}
 2       u.color = WHILE
 3       u.d = ∞
 4       u.π = NIL
 5  s.color = GRAY
 6  s.d = 0
 7  s.π = NIL
 8  Q = ∅
 9  ENQUEUE(Q, s)
10  while Q ≠ ∅
11       u = DEQUEUE(Q)
12       for each v ∈ G.Adj[u]
13            if v.color == WHITE
14                 v.color = GRAY
15                 v.d = u.d + 1
16                 v.π = u
17                 ENQUEUE(Q, v)
18       u.color = BLACK
```

Complexity: Using an adjacency list, for each vertex, $O(1)$ to enqueue it and $O(d(v))$ to examine all its neighbors. $T(n) = \sum_{v \in V} O(1 + d(v)) = O(V + \sum_{v \in V} d(v)) = O(V + 2E) = O(V + E)$.

The graph $(V, \{(v, v.\pi)\}_{v \in V})$ is a forest called the **BFS forest**. If all nodes are reachable from s, then the forest has one tree. Each tree in the forest is connected, and each node has a unique path to the root of the tree. Say the nodes with $v.d = L$ are at **layer** L of the BFS tree. All these nodes are **distance** L from the **source node** s.

9.4 DFS

Visit unvisited neighbors of a vertex when possible, otherwise **backtrack** to its parent. Use **timestamps** to track when events occur. The timestamps are used in many other algorithms using DFS as a subroutine.

Each node v has four fields:

$v.colot \in \{\text{WHITE}, \text{GRAY}, \text{BLACK}\}$ showing whether v is **undiscovered**, **being processed** or **processed**

$v.\pi$ giving v's parent

$v.d$ giving the time when v is **discovered**

$v.f$ giving the time when v is **finished**, i.e., itself and its non-parent neighbors have been processed

DFS(G)

```
1  for each vertex u ∈ G.V
2      u.color = WHITE
3      u.π = NIL
4  time = 0
5  for each vertex u ∈ G.V
6      if u.color == WHITE
7          DFS-VISIT(G, u)
```

DFS-VISIT(G, u)

```
1  time = time + 1              // white vertex u has just been discovered
2  u.d = time
3  u.color = GRAY
4  for each v ∈ G.Adj[u]        // explore edge (u, v)
5      if v.color == WHITE
6          v.π = u
7          DFS-VISIT(G, v)
8  u.color = BLACK              // blacken u; it is finished
9  time = time + 1
10 u.f = time
```

Complexity: Using an adjacency list, similar as BFS, $O(V + E)$.

The graph $(V, \{(v, v.\pi)\}_{v \in V})$ is a forest called the **DFS forest**.

Nesting property of discovery and finish time

u is a descendant of $v \implies [u.d, u.f] \subset [v.d, v.f]$

u and v are not descendants of each other $\implies$ $[u.d, u.f]$ and $[v.d, v.f]$ are **disjoint**

Classify each edge $e = (u, v) \in E$ as

tree edge (e is in the DFS tree)

back edge (v is an ancestor of u in the DFS tree)

forward edge (v is a descendant of u in the DFS tree)

cross edge (otherwise)

9.5 Applications of BFS and DFS

9.5.1 Connected components

Start with an arbitrary node s in V and run **BFS** or **DFS** from s. Let V' be all the black nodes. V forms one connected component. If $V - V' \neq \emptyset$, set $V = V - V'$, and repeat the above process to find more connected components.

Complexity: $O(V + E)$. Each node v is processed once, i.e., takes $O(1 + d(v))$ time.

9.5.2 Testing bipartiteness

```
IS-BIPARTITE(G)
1  s = any vertex v ∈ G.V
2  BFS(G, s)
3  for each vertex u ∈ G.V
4       if u.d is even
5            u.color = RED
6  for each edge (u, v) ∈ G.E
7       if u.color == v.color
8            return FALSE
9  return TRUE
```

Complexity: $O(V + E)$

9.5.3 Satisfiability (2SAT)

Write expressions in **conjunctive normal form** (CNF), i.e., as the ANDs of ORs. In k-CNF, each clause has k literals. Brute force solution $O(2^k)$.

Given a 2-CNF formula ϕ, convert it into a directed graph G. There are two nodes, called x and $\neg x$, for each variable x in ϕ. For each clause of the form $(\neg a \vee b)$ in ϕ, create an edge (a, b).

If theres a path from a node u to v, then theres also a path from $\neg v$ to $\neg u$.

Suppose there is a variable x such that there is a path from x to $\neg x$ and a path from $\neg x$ to x in G, Then ϕ is not satisfiable.

```
2-SAT(G)
 1  for each vertex x ∈ G.V
 2       x.value = NIL
 3       if EXIST-PATH(x, ¬x) and EXIST-PATH(¬x, x)
 4            return FALSE
 5  Q = G.V
 6  while Q ≠ ∅
 7       x = any vertex such that EXIST-PATH(x, ¬x) == FALSE
 8       R = {all the vertices reachable from x found by BFS or DFS}
 9       for each vertex v ∈ R
10            if v.value == NIL
11                 v.value = TRUE
12                 Q = Q − {v}
13            if ¬v.value == NIL
14                 ¬v.value = FALSE
15                 Q = Q − {¬v}
16  return TRUE
```

Complexity: $O(n(m + n))$, where n, m are the number of literals and clauses, resp.

9.5.4 8 queens puzzle

- Use a directed graph to search for solutions.
- Each vertex represents a legal placement of $k \leq 8$ queens. The root of the tree represents the empty board. At most 8^8 nodes, actually far less.

- Edge (u, v) means placement v is an extension of u.
- Nodes on level 8 of solution graph are solutions to puzzle.
- Algorithm
  - Run DFS, starting from the root node (empty board).
  - If current node has $k > 0$ child nodes, run DFS from each child node in order. After all DFS's finished, backtrack to parent node.
  - If node were exploring has 0 child nodes, backtrack to parent.
  - If we reach a level 8 node, record it as a solution.
- Uses $O(1)$ memory. Only need to keep track of the parent nodes in the DFS branch were searching.
- Huge time complexity since DFS needs to go through all nodes of the solution tree, but still much faster than brute force search.

9.5.5 Solving a maze

- Represent the maze by an undirected graph.
- Nodes are **junctions**, where multiple paths branch off. Also two nodes for the start and finish.
- Edges between nodes that dont have intermediate junctions.
- Algorithm
  - Run DFS on graph from the start node.
  - If we visit finish node, chain of parent nodes back to start node is maze solution.
  - If DFS returns without visiting finish node, maze has no solution.

9.5.6 Partial orders

- A **partial order** is an (incomplete) ordering on a set.
- Can be represented by a directed graph, where vertices are the elements in the set, and an edge (v, w) means $v < w$ in the order.
- Graph must be **acyclic** for the partial order to be valid, called a **directed acyclic graph** (DAG).
- **Topological sort** of the DAG representing the partial order: Find a **total order** consistent with the partial order. There may be multiple consistent orderings, we can return any one.
- Suppose $v < w$ in a partial order corresponding to G. Then $v.f > w.f$.

TOPOLOGICAL-SORT(G)
1 call DFS(G) to compute finishing times $v.f$ for each vertex v
2 as each vertex is finished, insert it onto the front of a linked list
3 **return** the linked list of vertices

Complexity: $O(V + E)$

9.5.7 Strongly connected components

- A maximal set of nodes C in a directed graph G such that for all $v, w \in C$, both paths from node v to w and from w to v exist.
- Suppose we contract all the nodes in each SCC of G to a single node. This forms a component graph G', then G' is a DAG.
- The SCCs output in the second DFS occur in the **topological sort order** of the component graph.

STRONGLY-CONNECTED-COMPONENTS(G)

1 call DFS(G) to compute finishing time $u.f$ for each vertex u
2 compute G^{T} // with all edges in G reversed
3 call DFS(G^{T}), but in the main loop of DFS, consider the vertices in order of decreasing $u.f$ (as computed in line 1)
4 output the vertices of each tree in the depth-first forest formed in line 3 as a separate strongly connected component

Completxity: $O(V + E) = 2O(V + E) + O(E)$

9.5.8 Biconnected components

- **Articulation point:** a node whose removal separates the graph into multiple parts.
- **Biconnected graph:** graph without articulation points, are either a single edge or have ≥ 2 **edge disjoing paths** between any pair of nodes.
- **Biconnected component:** a maximal biconnected subgraph.

BICONNECTED-COMPONENTS(G)

```
1  S = ∅        // S is a stack different from the one used to implement DFS itself
2  C = ∅        // C is the set of edges in a biconnected component
3  SCC = ∅
4  during DFS(G)
5      for each edge e encountered
6          S.push(e)
7      if DFS backtracks to an edge (u, v) where u is an articulation point
8          repeat
9              e = S.pop()
10             C = C ∪ {e}
11         until e == (u, v)
12         SCC = SCC ∪ {C}
13         C = ∅
14 return SCC
```

Complexity: $O(V + E)$ if the articulation points are known.

9.6 MST

Start with no edges, and add one edge per stage. At every stage, edges form a **partial MST**, which does not have any **cycles** and is a **forest**. Stop when adding any edge creats a cycle.

MST principle: Let G be a weighted graph. Suppose H is a **subset of some MST** of G. C is **any connected component** of H. S is set of edges connecting C to **any other component** of H. e is the **min weight edge** in S. Then $H \cup \{e\}$ is also a **subset of some MST** of G.

9.6.1 Prim's algorithm

MST-PRIM(G, w)

```
1  // H contains the nodes in the partial MST
2  H = {an arbitrary node v ∈ G.V}
3  // F contains the edges in the partial MST
4  F = ∅
5  // R contains the edges touching F and not creating a cycle, initially containing all edges from G.V
6  R = ∅
7  for each vertex u ∈ G.Adj[v]
8      R = R ∪ {(u, v)}
9  while H ≠ G.V
10     e = GET-MIN-WEIGHT-EDGE(R)
11     let e = (u, v), with v ∉ H
12     H = H ∪ {v}
13     F = F ∪ {e}
14     for each vertex u ∈ G.Adj[v]
15         if u ∉ H
16             R = R ∪ {(u, v)}
```

Complexity: $O(E \log V) = O(1) + O(V) + O(V \log V) + O(E \log V)$, R implemented by priority queue.

9.6.2 Kruskal's algorithm

MST-KRUSKAL(G, w)

```
1  A = ∅
2  for each vertex v ∈ G.V
3      MAKE-SET(v)
4  sort the edges of G.E into nondecreasing order by weight w
5  for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
6      if FIND-SET(u) ≠ FIND-SET(v)
7          A = A ∪ {(u, v)}
8          UNION(u, v)
9  return A
```

Complexity: $O(E \log V) = O(V) + O(E \log V) + EO(\log V)$

9.6.3 Arborescences

- **Directed spanning trees**. Given a directed graph and a root vertex r, there is a unique directed path from r to every other vertex.
- BFS or DFS can find some arborescence, if one exists.
- A subgraph T of G is an arborescence rooted at node $r \iff T$ has no directed cycles and every node $v \neq r$ has exactly one incoming edge
- **Edge reweighting**
  - For each vertex v, let $\delta(v)$ be the minimum cost of any incoming edge.
  - For each of its incoming edges (u, v), set $w^*(u, v) = w(u, v) - \delta(v)$.
  - Call the reweighted graph G'. All edge weights in G' are nonnegative.

- An arborescence is minimum weight for G' if and only if it is minimum weight for G.
- Each vertex in G' has at least one incoming edge with weight 0.
- If a set of weight 0 edges form an arborescence, then it has min weight.

ARBORESCENCE(G, w)

```
1  for each vertex v ∈ G. V
2        v.δ = min(weight of incoming edge (u, v))
3        for each incoming edge (u, v)
4              w(u, v) = w(u, v) − v.δ
5  Q = ∅
6  for each vertex v ∈ G. V
7        Q = Q ∪ {one of (u, v) such that w(u, v) = 0}
8  if Q is an arborescence
9        return Q
10 C = a directed cycle found by BFS or DFS
11 v = CONTRACT(C)
12 return ARBORESCENCE(G − C + {v}, w) + C − {one edge of C}
```

Complexity: $O(VE)$

9.7 Single source shortest paths (SSSP)

9.7.1 Relaxations

INITIALIZE-SINGLE-SOURCE(G, s)

```
1  for each vertex v ∈ G. V
2        v. d = ∞
3        v.π = NIL
4  s. d = 0
```

RELAX(u, v, w)

```
1  if v. d > u. d + w(u, v)
2        v. d = u. d + w(u, v)
3        v.π = u
```

9.7.2 Negative weight cycles

- A cycle in the graph such that the sum of all weights on the cycle is negative.
- If a graph has a negative weight cycle reachable from the source, then shortest paths are not well defined, because we can repeatedly go around the cycle to get arbitrarily short paths.

9.7.3 Properties of shortest paths

- Suppose a graph doesnt contain any negative weight cycles, and all nodes are reachable from the source s.
- **Triangel inequality:** The length of a shortest path from s to v, $\delta(s, v) \leq \delta(s, u) + w(u, v)$
- **Shortest subpaths:** Let $p = \langle (s, v_0), (v_0, v_1), \ldots, (v_k, v) \rangle$ be a shortest path from s to some node v, then for every $v_i \in \{v_0, \ldots, v_k\}$, $\langle (s, v_0), (v_0, v_1), \ldots, (v_{i-1}, v_i) \rangle$ is a shortest path from s to v_i.

- **Shortest path tree:** There exists a tree T rooted at s such that the shortest path from s to any node v lies in T. Each node v has a parent in the tree. By following parent pointers starting from v, we find shortest path from s to v.

9.7.4 Bellman-Ford algorithm

Returns FALSE if there is a negative weight cycle reachable from s in the graph. Settles nodes in **order of number of edges** on the shortest path.

```
BELLMAN-FORD(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  for i = 1 to |G.V| − 1
3       for each edge (u, v) ∈ G.E
4            RELAX(u, v, w)
5  for each edge (u, v) ∈ G.E
6       if v.d > u.d + w(u, v)
7            return FALSE
8  return TRUE
```

Complexity: $O(VE)$

9.7.5 Dijkstra's algorithm

Assumes all weights are nonnegative. Settles nodes in **order of distance** of node from source. Q is a priority queue.

```
DIJKSTRA(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  S = ∅
3  Q = G.V
4  while Q ≠ ∅
5       u = EXTRACT-MIN(Q)
6       S = S ∪ {u}
7       for each vertex v ∈ G.Adj[u]
8            RELAX(u, v, w)
```

Complexity: $O(E \log V)$ using a binary (min) heap, $O(V \log V + E)$ using a Fibonacci heap

9.8 All pairs shortest paths (APSP)

- Want to find the shortest paths from each node to all the other nodes.
- Represent this by an $n \times n$ matrix L, where $L_{i,j}$ is the length of shortest path from i to j, and $n = |V|$.
- Can also output a predecessor matrix Π, where $\Pi_{i,j}$ is the last node before j on the shortest path from i.
- Assume in following that there are no negative weight cycles. Algorithms can be modified to detect negative weight cycles.
- Can solve APSP by running SSSP n times.
  - If all weights are nonnegative, running Dijkstra $|V|$ times takes $O(V^2 \log V + VE)$ time.
  - For general weights, can run Bellman-Ford $|V|$ times, in $O(V^2E) = O(V^4)$ time.

9.8.1 Recursive shortest paths

- Given nodes i, j, and $1 \leq m < n$, let $l_{i,j}^{(m)}$ be the length of the shortest path from i to j using at most m edges, or ∞ if there is no such path. Set $l_{i,j}^{(0)} = 0$ for $i = j$, and $l_{i,j}^{(0)} = \infty$ otherwise.
- $l_{i,j}^{(m)} = \min\limits_{1 \leq k \leq n} \{l_{i,k}^{(m-1)} + w(k, j)\}$
- The shortest path distances are given by $\{l_{i,j}^{(n-1)}\}_{i,j}$

9.8.2 Matrix multiplication APSP

EXTEND-SHORTEST-PATHS(L, W)

```
1  n = L.rows
2  let L' = (l'_ij) be a new n × n matrix
3  for i = 1 to n
4      for j = 1 to n
5          l'_ij = ∞
6          for k = 1 to n
7              l'_ij = min(l'_ij, l_ik + w_kj)
8  return L'
```

APSP(W)

```
1  n = W.rows
2  L^(1) = W
3  m = 1
4  while m < n − 1
5      let L^(2m) be a new n × n matrix
6      L^(2m) = EXTEND-SHORTEST-PATHS(L^(m), L^(m))
7      m = 2m
8  return L^(m)
```

Complexity: similar to matrix multiplication, $O(n^3 \log n)$ using **repeated doubling**

9.8.3 Floyd-Warshall algorithm

Consider the shortest path from i to j using only nodes $1, \ldots, k$ as **intermediate nodes**, which are nodes on the path besides i and j.

Let $d_{i,j}^{(k)}$ be the minimum distance from i to j using $1, \ldots, k$ as intermediate nodes. The shortest path distance from i to j equals $d_{i,j}^{(n)}$. Define $d_{i,j}^{(0)} = w_{i,j}$.

FLOYD-WARSHALL(W)

```
1  n = W.rows
2  D^(0) = W
3  for k = 1 to n
4      let D^(k) = (d_ij^(k)) be a new n × n matrix
5      for i = 1 to n
6          for j = 1 to n
7              d_ij^(k) = min(d_ij^(k−1), d_ik^(k−1) + d_kj^(k−1))
8  return D^(n)
```

Complexity: $O(n^3)$, can also compute the predecessor matrix in $O(n^3)$ time.

9.8.4 Johnson's algorithm

JOHNSON(G, w)

```
 1  compute G', where G'.V = G.V ∪ {s}, G'.E = G.E ∪ {(s, v) : v ∈ G.V}, and w(s, v) = 0 for all v ∈ G.V
 2  if BELLMAN-FORD(G', w, s) == FALSE
 3      print "the input graph contains a negative-weight cycle"
 4  else
 5      for each vertex v ∈ G'.V
 6          set h(v) to the value of δ(s, v) computed by the Bellman-Ford algorithm
 7      for each edge (u, v) ∈ G'.E
 8          ŵ(u, v) = w(u, v) + h(u) − h(v)
 9      let D = (d_uv) be a new n × n matrix
10      for each vertex u ∈ G.V
11          run DIJKSTRA(G, ŵ, u) to compute δ̂(u, v) for all v ∈ G.V
12          for each vertex v ∈ G.V
13              d_uv = δ̂(u, v) + h(v) − h(u)
14      return D
```

Complexity: $O(V^2 \log V + VE) = O(VE) + O(V^2 \log V + VE)$